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ABELIAN AND NILPOTENT SUBGROUPS OF MAXIMAL ORDERS OF FINITE SIMPLE GROUPS

Phd thesis

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Contents

Chapter 1

Introduction

§1 General characteristic of the paper

After announcement on the classification of finite simple groups investigation of known simple groups becomes one of the most important problem in finite group theory. In particular, subgroups structure of known finite simple group is of interest. The most important subgroups are maximal subgroups, maximal soluble subgroups, maximal nilpotent subgroups, and maximal abelian subgroups. This paper is devoted to abelian and nilpotent subgroups of maximal order of finite simple groups.

The main part of known finite simple groups consists of finite simple groups of Lie type. This groups are divided into 16 classes. Six classes are so-called classical groups and ten classes are so-called exceptional groups. Since the subgroups structure of this groups is the most complicated, the most part of the paper dedicated to these groups. Because of closed relation between finite groups of Lie type and simple linear algebraic groups defined over algebraically closed field of positive characteristic we obtain some auxiliary results for algebraic groups.

Recall some known results on structure and orders of some special subgroups of finite simple groups of Lie type. The structure of Sylow p-subgroups, where p is the characteristic of base field, is found by Chevalley. So finite groups of Lie type are often called Chevalley groups. Carter, Weir, and Wong in [13], [42], and [47] find structure of Sylow r-subgroups, where r does not equal to the characteristic of base field. In 60-s many authors investigate structure and orders of maximal tory in finite Chevalley groups. In 1972 Carter [11] find simple and clear way to finding orders of maximal tory in all split finite Chevalley groups. From 1979 till 1982 some papers written by Barry and Wong [3], [4], [45], and [46] are published. They find p-ranks, Thompson subgroups and abelian unipotent subgroups of maximal order in all finite classical groups. The analogous problem for exceptional groups remains open. This problem, in particular may be found in [28].

In the work we find orders, structure and classes of abelian and nilpotent subgroups of maximal order in almost all finite simple groups. More over, we find abelian and nilpotent subgroups of maximal order in symmetric groups and finite groups of Lie type (not necessary simple). We find abelian unipotent subgroups of maximal order, p-ranks, and Thompson subgroup in all maximal unipotent subgroups of finite Chevalley groups. Thus, we solve the problem, mentioned above. We also prove that abelian subgroup of maximal order in finite simple group of Lie type equals either some tory or some abelian unipotent subgroup of maximal order. Orders and structure of abelian subgroups in

symmetric and alternating groups are found by direct calculation. We use ATLAS [15] to find abelian subgroups of maximal order on finite symplectic groups. Thus, in the work orders, structure and classes of abelian subgroups of maximal order are found in all finite simple groups, except some symplectic groups of large order. We obtain following theorem as a corollary.

Theorem 1.1.1. *Let G be a nonabelian finite group such that $G \not\cong A_1(q)$ and let A be an abelian subgroup of G. Then $|A|^3 < |G|$.*

The following theorem gives a solution to the problem 4.27 from "Kourovka Notebook" [49]. A solution to this problem in another way was announced earlier in [23].

Theorem 1.1.2. *A nonabelian finite simple group G has a decomposition ABA, where A, B are abelian subgroups of G if and only if $G \cong L_2(2^n)$ for some $n \geqslant 2$. Moreover $|A| = 2^t + 1$, $|B| = 2^t$, A is a cyclic group, and B is elementary abelian.*

We prove that nilpotent subgroups of maximal order in almost all finite simple groups coincide with some Sylow subgroup. We also prove that there exist just one class of large nilpotent subgroups in all finite simple groups. The following theorem generalizes result by Mazurov and Zenkov [34].

Theorem 1.1.3. *Let G be a nonabelian finite simple group, N a nilpotent subgroup of G. Then $|N|^2 < |G|$.*

Note that soluble subgroups of maximal order in all finite simple groups (except sporadic) are found in [33] and [38].

Let Ψ be a some property (for example, commutativity, nilpotency, solvability, etc.) We consider the following problem.

Problem. If a finite group G contains a Ψ-subgorup of index n, then is it true that G contains a normal Ψ-subgroup of index $\leqslant f(n)$ for some function $f(n)$.

Clearly, that for an arbitrary property Ψ such that Ψ is inherited by all subgroups we may take $f(n) = n!$. So we demand that $f(n) = n^c$ for some constant c. Note that if Ψ is solvability or cyclicity, then an affirmative answer is obtained in [2]. In this paper they also formulate the question mentioned above when Ψ is commutativity or nilpotency. When Ψ is commutativity Muzychuk find an affirmative answer with function $f(n) = n^2$ in unpublished paper. In this paper we get an affirmative answer in remaining case, when Ψ is nilpotency (with function $f(n) = n^9$).

Since results by Babai, Goodman, and Pyber allow us to reduce problem to soluble groups, we prove the following theorem.

Theorem 1.1.4. *Let G be a finite soluble group, containing soluble subgroup of index n. Then $|G : F(G)| < n^5$.*

The work consists of six chapters (with introduction). In the first chapter we give basic definitions and known results.

In the second chapter we find abelian subgroups of maximal order in symmetric, alternating, sporadic groups, and maximal unipotent subgroups of finite Chevalley groups of exceptional type. In maximal unipotent subgroups we also find p-ranks and Thompson subgroups. For convenience we assemble the results of the second chapter in tables 2.1,

2.4, and 2.5. The results of this chapter were announced at International Science Student Conference (ISSC) (see [51], [54]), at Krasnoyarsk Conference (see [53]), at International algebraic conference dedicated to the memory of D. K. Faddeev in St.Petersburg (see [52]), at International algebraic conference “Groups and Group Rings” in Poland (see [57]), and at International algebraic conference dedicated to the memory of Yu. I. Merzlyakov in Novosibirsk (see [58]). The results have been published in [59] and [63]. Note, that Barry and Wong in papers [3], [4], [45], and [46] use canonical rational representation to find abelian unipotent subgroups of maximal order in finite classical groups. Theorems 2.2.1, 2.2.2, and 2.2.4 give simple and clear way to finding disconjugate unipotent abelian subgroups of maximal order in all finite split groups of Lie type. More over, this results may be easily generalized for reductive linear algebraic groups over field of arbitrary characteristic. In particular, one can derive results by Barry and Wong for split classical groups by using this theorems. We make a GAP program to find maximal abelian with respect to p subsets in all exceptional root systems. We give complete lists of maximal abelian with respect to p (for different p) in root systems G_2, D_4, and F_4.

In the third chapter we prove that the order of an arbitrary abelian subgroup of a finite simple group of Lie type is less then or equal to the order of some semisimple or some unipotent abelian subgroup. In particular, we find abelian subgroups of maximal order in all (not only simple) finite Chevalley groups. We assemble results of this chapter in the table 3.2. The results of the third chapter were announced at ISSC (see [54]), at International algebraic conference dedicated to the memory of A. G. Kurosh in Moscow, at International algebraic conference “Groups and Group Rings” in Poland (see [57]), and at International algebraic conference dedicated to the memory of Yu. I. Merzlyakov (see [58]). The results have been published in [59], and [61]. We use canonical rational representation when we investigate groups $A_n(q)$ and $C_n(q)$, and we use linear algebraic groups technique in other cases. The structure of ${}^2B_2(q)$ and ${}^2G_2(q)$ may be find in [40] and [41], so we give orders of abelian subgroups of maximal order in this groups without any arguments.

In the forth chapter we find orders, structure, and number of classes of nilpotent subgroups of maximal order in all finite simple groups. We also prove theorem 1.1.3. The results of this chapter were announced at ISSC (see [56]), at International algebraic conference “Groups and Group Rings” in Poland (see [57]), and at International algebraic conference dedicated to the memory of Yu. I. Merzlyakov in Novosibirsk (see [58]). This results have been published in [62]. For convenience we assemble the results of the forth chapter in the tables 4.1 and 4.3.

In the fifth chapter we prove theorem 1.1.4. The results were announced at ISSC (see [56]), at International algebraic conference dedicated to the memory of A. G. Kurosh in Moscow (see [55]), at International algebraic conference “Groups and Group Rings” in Poland (see [57]), and at International algebraic conference dedicated to the memory of Yu. I. Merzlyakov in Novosibirsk (see [58]). The results have been published in [60].

In the six chapter we obtain some corollaries of previous results. We prove theorem 1.1.2 and find an affirmative answer to the Babai, Goodman, and Pyber problem. The results were announced at ISSC (see [51] and [56]), at Krasnoyarsk conference (see [53]), at International algebraic conference dedicated to the memory of D. K. Faddeev in St.Petersburg (see [52]), at International algebraic conference dedicated to the memory of A. G. Kurosh in Moscow (see [55]), at International algebraic conference “Groups and Group Rings” in Poland (see [57]), and at International algebraic conference dedicated

to the memory of Yu. I. Merzlyakov in Novosibirsk (see [58]). The results have been published in [59] and [60].

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§2 Group theory notations

Definitions and notations of the present paper can be found in [37]. For a group G denote by $H \leq G$ and $H \trianglelefteq G$ a subgroup and a normal subgroup H of G respectively. We define $|G : H|$ to be the index of H, $N_G(H)$ to be the normalizer of H. If $H \trianglelefteq G$, then we denote by G/H the factor group of H in G. Suppose M is a subset of G. Denote by $\langle M \rangle$ the subgroup of G, generated by M and by $|M|$ the cardinality of M (or the order of the element if M consists of just one element). By $C_G(M)$ we denote the centralizer of M in G. Let $Z(G)$ be the center of G. A conjugation of an element x by an element y is denoted x^y and defined by $x = y^{-1}xy$. The element $[x, y] = x^{-1}x^y$ is called the commutator of x and y. We denote the commutant of subgroups A and B of G by $[A, B]$. Suppose that A and B are groups. Then symbols $A \times B$, $A * B$, and $A \leftthreetimes B$ denote direct, commutative, and semidirect products of A and B with normal subgroup B respectively. If A and B are subgroups of G such that $B \trianglelefteq A \leq G$, then the factor group A/B is called *a section* of G. Symbols $\Phi(G)$ and $F(G)$ denote Frattini and Fitting subgroups of G respectively.

By $\mathrm{Syl}_p(G)$ we denote the set of all Sylow p-subgroups of G. Let φ be a homomorphism of G and let g be in G. Then symbols G^φ and g^φ denote images of G and g respectively. If φ is an endomorphism of G we denote the set of φ-stable points of G by G_φ. $\mathrm{Aut}\, G$ and $\mathrm{Out}\, G$ denote automorphism group and outer automorphism group of G respectively.

Let ω be a set of primes. By ω' we define the set of all primes not in ω. If the set ω consists of just one element, we will use just a p' notation. Recall that $O_\omega(G)$ is the maximal normal ω-subgroup of G and $O^\omega(G)$ is a subgroup generated by the set of ω'-elements. For a prime p symbol $i_p(G)$ denotes the minimal integer k such that the intersection of k Sylow p-subgroups equals $O_p(G)$.

Definition 1.2.1. An abelian (resp. nilpotent) subgroup of maximal order of finite group G is said to be *large*.

The set of all abelian (resp. elementary abelian and normal abelian) subgroups of maximal order of finite group G is denoted by $\mathbf{A}(G)$ (resp. by $\mathbf{A}_e(G)$ and $\mathbf{A}_n(G)$). The order of some element of $\mathbf{A}(G)$ (resp. $\mathbf{A}_e(G)$ and $\mathbf{A}_n(G)$) is denoted by $\mathbf{a}(G)$ (resp. $\mathbf{a}_e(G)$ and $\mathbf{a}_n(G)$). By $J(G)$, $J_e(G)$, and $J_n(G)$ we denote subgroups generated by elements of $\mathbf{A}(G)$, $\mathbf{A}_e(G)$, and $\mathbf{A}_n(G)$ respectively. The set of all nilpotent subgroups of maximal order is denoted by $\mathbf{N}(G)$, the order of some element of $\mathbf{N}(G)$ by $\mathbf{n}(G)$. By $\mathbf{c}_a(G)$ (resp. $\mathbf{c}_n(G)$) we denote the number of classes of large abelian (resp. nilpotent) subgroups. The symbol $m_p(G)$ denotes p-rank of G, i. e. the maximum of ranks of abelian p-subgroups.

§3 The structure of finite groups of Lie type

Notations and results of finite Chevalley groups can be found in [12]. By Chevalley groups we mean universal Chevalley groups and its central factors. By $GF(q)$ we denote the finite field of order q, by p its characteristic, by $GF(q)^*$ multiplicative group of $GF(q)$. A Chevalley group G with root system Φ over field $GF(q)$ is denoted by $\Phi(q)$. The field $GF(q)$ is called *base* field of G. The Weil group of root system Φ is denoted by $W(\Phi)$. The Weil group of G is denoted by $W(G)$.

If G is a split finite group of Lie type, then X_r is a root subgroup, where $r \in \Phi$. X_r is known to be isomorphic with additive group of $GF(q)$. By $x_r(t)$ is denoted an element of X_r, where t is the image of $x_r(t)$ under above isomorphism. Every Chevalley group G is generated by its root subgroups X_r $(r \in \Phi)$.

Define a *graph automorphism* of a finite group of Lie type G. Assume, that Dynkin diagram of G admits a symmetry ρ such that this symmetry can be resumed to an automorphism φ of G. Then $x_r(t)^\varphi = x_{r^\rho}(t^\lambda)$, where λ is the field automorphism, corresponding to φ, and φ can be resumed to G in natural way. We denote r^ρ by $\overline{r}$, r^{ρ^2} by $\overline{\overline{r}}$, where $r \in \Phi$. The image of an element $t \in GF(q^\alpha)$ ($\alpha = 1$, 2 or 3) under λ is denoted by $\overline{t} = t^\lambda$, $\overline{\overline{t}} = t^{\lambda^2}$ respectively. The group $O^{p'}(G_\varphi)$ is called *twisted* groups of Lie type.

Twisted groups are denoted by ${}^2A_n(q^2)$, ${}^2D_n(q^2)$, ${}^2E_6(q^2)$, ${}^3D_4(q^3)$, ${}^2B_2(q)$, ${}^2G_2(q)$, and ${}^2F_4(q)$. If G is isomorphic to ${}^2A_n(q^2)$, ${}^2D_n(q^2)$, ${}^2E_6(q^2)$, then we say that $GF(q^2)$ is the *base* field of G. If G is isomorphic to ${}^3D_4(q^3)$, then $GF(q^3)$ is the *base* field of G. $GF(q)$ is the *base* field of G in remaining cases. The field $GF(q)$ is called the *definition field* for all groups of Lie type.

Definition 1.3.1. A p-element x of a Chevalley group $\Phi(q)$ is called *unipotent*, a p'-element is called *semisimple*. We define semisimple and unipotent subgroups of G in similar way. The set of all semisimple (resp. unipotent) abelian subgroups of maximal order is denoted by $\mathbf{A}_s(G)$ (resp. $\mathbf{A}_u(G)$). The set of all semisimple nilpotent subgroups of maximal order is denoted by $\mathbf{N}_s(G)$. Note, that we do not need to define the set $\mathbf{N}_u(G)$, since $\mathbf{N}_u(G) = \mathrm{Syl}_p(G)$.

Recall, that for every root system Φ there exists a set of roots $r_1, \ldots, r_n$ such that every root is uniquely decomposed as $\sum_{i=1}^n \alpha_i r_i$, where all α_i are integers and either nonnegative, or nonpositive. Such set of root is called *fundamental system* of Φ. A fundamental system is a basis of $\mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R}$. The dimension of $\mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R}$ is called *rank* of root system Φ. We assume below, that all fundamental roots are positive. Then a root r is positive if and only if it equals to $\sum_{i=1}^n \alpha_i r_i$ and all α_i are nonnegative. For a root system Φ we denote by Φ^+ (resp. Φ^-) the set of positive (resp. negative) roots. If $r = \sum_{i=1}^n \alpha_i r_i$ then the number $\sum_{i=1}^n \alpha_i$ is called *weight* of r and is denoted by $h(r)$. In every indecomposable root system Φ there exists a unique root of highest weight and this root is denoted by r_0.

We will say that a root system Φ (or a space $\mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R}$) has a partial (or linear) order if this order agrees with addition and multiplication on real number. We assume below, that $\mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R}$ has a linear order $\leqslant$, defined by $0 \leqslant v = \sum_{i=1}^n \alpha_i r_i$ if and only if either $v = 0$, or last nonzero coefficient α_i is greater then 0. The notation $r \leqslant s$ means that $0 \leqslant s - r$ and the notation $r < s$ means $r \leqslant s$ and $r \neq s$. Moreover, we fix a partial order $\preceq$ defined by $r \preceq s$ if and only if $s - r = \sum_{i=1}^n \alpha_i r_i$ and all α_i are nonnegative.

If G is a finite Chevalley group with the base field $GF(q)$ of characteristic p, then an element of $\mathrm{Syl}_p(G)$ is denoted by U. Clearly we may assume that $U = \langle X_r | r \in \Phi^+ \rangle$.

If a root system Φ has some linear order, then every element is known to be uniquely decomposed into the product of $x_r(t)$ taken in given order. The following lemma called by Chevalley commutator formulae is known.

Lemma 1.3.2. *[12, 5.2.2] Let $x_r(t)$, $x_s(u)$ be elements of X_r and X_s respectively and $r \neq s$. Then*

$$[x_r(t), x_s(u)] = \prod_{ir+js\in\Phi; i,j>0} x_{ir+js}(C_{ijrs}(-t)^i u^j),$$

where C_{ijrs} do not depend on t and u.

This formulae means, that the commutant of X_r and X_s is in X_{ir+js}, where $i, j > 0$ and $ir + js \in \Phi$.

Definition 1.3.3. Let p be the characteristic of the base field of a finite Chevalley group G. A subset Ψ of the root system Φ is called *abelian with respect to p* if for every roots $r, s \in \Psi$ either $r + s$ not in Φ, or $C_{11rs} = 0$ in characteristic p. For a root system Φ the maximum of orders of abelian with respect to p subsets of Φ^+ is denoted by $\mathbf{a}(\Phi, p)$. A subset Ψ of a root system Φ is called *abelian* if for every $r, s \in \Psi$ the vector $r + s$ is not in Φ. Abelian subsets of maximal orders in all irreducible root systems are found by A. I. Mal'tsev in [32].

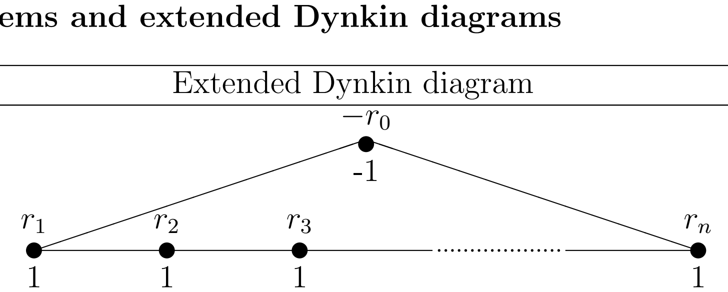
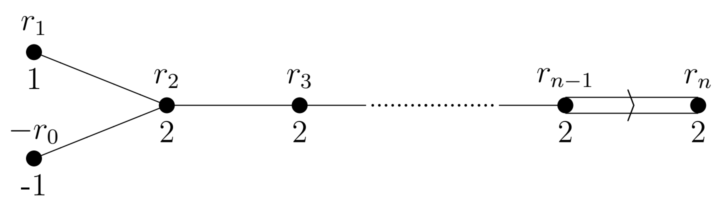
Definition 1.3.4. Let $x \in U$ be a unipotent element, then the decomposition $x = \prod_{r\in\Phi^+} x_r(t_r)$, where $t_r \in GF(q)$ and roots r agree with the order $\leqslant$, is called *canonical form* of x. Some of t_r prove to be equal zero.

Every unipotent element x of U has a unique canonical form. The set of roots $\{r\}$ such that $t_r \neq 0$ is denoted by $\Phi(x)$. If $L \leq U$ is a unipotent subgroup of G, then $\Phi(L) = \bigcup_{x\in L} \Phi(x)$ by definition. By $m(x)$ we define the minimal element of $\Phi(x)$. If $L \leq U$, then $m(L) = \bigcup_{x\in L}\{m(x)\}$ by definition.

Definition 1.3.5. Add the root $-r_0$ to Dynkin diagram and join it to other nodes by usual rule. We derive *extended Dynkin diagram*

In the table 1.1 we assemble information about extended Dynkin diagram for all irreducible root systems. The indexing in the table agrees with the one given in [16].

Table 1.1. Root systems and extended Dynkin diagrams

<table>
<tr><th>Type Φ</th><th>Extended Dynkin diagram</th></tr>
<tr><td>A_n</td><td>
</td></tr>
<tr><td>B_n</td><td>
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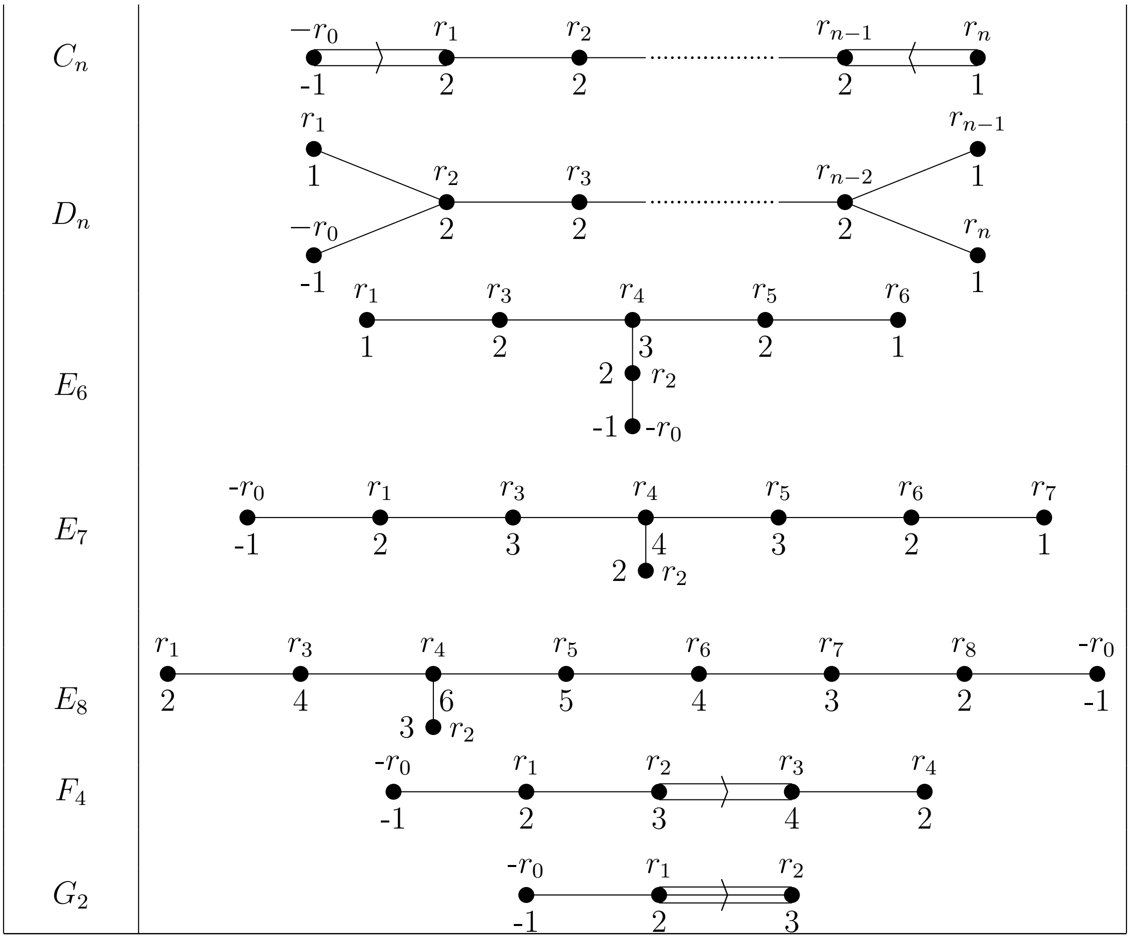

§4 Linear algebraic groups

The information about structure and properties of linear algebraic groups can be found in [20]. Since we deal with linear algebraic groups only, we omit word "linear" below.

If G is an algebraic group, then we denote the component of the unite of G by G^0. An algebraic group is said to be semisimple if its radical is trivial, an algebraic group is said to be reductive if its unipotent radical is trivial. In both cases we do not assume G to be connected. A connected semisimple algebraic group is known to be the commutative product of connected simple algebraic groups and a connected reductive algebraic group is known to be the commutative product of torus S and semisimple connected group M, where $S = Z(G)^0$, $M = [G, G]$, and $S \cap M$ is finite. Note that this definition do note agrees with the definition 1.3.1. Thus if we consider semisimple subgroup of algebraic group we use just given definition. For finite simple groups of Lie type we use definition 1.3.1.

Let G be a connected reductive algebraic group and T some maximal torus of G. The dimension of T is called the *rank* of G.

Every element of connected reductive group G is known to be uniquely decomposed into the product un_wtv, where $v \in U$, $t \in T$, $u \in U \cap n_wU^-n_w^{-1}$ (see [20, theorem 28.3], for instance). Here n_w is some element of complete preimage of w under natural homomorphism $N_G(T) \to W$. Such a decomposition is called *Bruhat* decomposition.

Let G be a connected algebraic group, let π be some rational faithful representation of G, let Γ_π be the lattice generated by weights of π. The lattice, generated by Φ is denoted

by Γ_{ad}, by Γ_{sc} we denote the lattice, generated by fundamental roots. The lattices Γ_{sc}, Γ_π, and Γ_{ad} do not depend on π and the following inclusions hold $\Gamma_{ad} \leq \Gamma_\pi \leq \Gamma_{sc}$.

For given root system Φ there exist several nonisomorphic connected simple algebraic groups. They are called *isogenies.* They have nonisomorphic groups Γ_π and nonisomorphic finite center. If $\Gamma_\pi = \Gamma_{sc}$, then G is called *simply connected* and is defined by G_{sc}. If $\Gamma_\pi = \Gamma_{ad}$, then G is of adjoint type and is defined by G_{ad}. Every connected algebraic group with root system Φ is a factor group G_{sc}/Z, where $Z \leqslant Z(G_{sc})$. The center of G_{ad} is trivial and G_{ad} is a simple abstract group.

Let $r_0 = \sum_i c_i r_i$ be the decomposition of r_0 into the sum of fundamental roots with integer coefficients. The numbers c_i may be found in the table 1.1. All prime dividers of this numbers are *bad* primes for a group G.

Lemma 1.4.1. *[20, 21.3 and 22.2] Let G be a connected linear algebraic group. Then all Borel subgroups are conjugate in G. More over, maximal tory and maximal unipotent subgroups of G coincide with maximal tory and maximal unipotent subgroups of Borel subgroups. All maximal tory and maximal unipotent subgroups of G are conjugate in G and every semisimple (resp. unipotent) element of G lies in some maximal tory (resp. maximal unipotent subgroup).*

Let $\overline{G}$ be a connected simple algebraic group over algebraically closed field of characteristic $p > 0$, let σ be an endomorphism of $\overline{G}$ such that the set $\overline{G}_\sigma$ is finite. Such an endomorphism σ is said to be Frobenius automorphism. although it might do not coincide with some classical Frobenius automorphism. Note, that σ is an automorphism if we suppose $\overline{G}$ to be an abstract group, and σ is an endomorphism if we suppose $\overline{G}$ to be an algebraic group. In general case σ equals $q\sigma_0$, where $q = p^\alpha$ is an exponentiation to the power q and σ_0 is a graph automorphism of order 1, 2, or 3. Then $O^{p'}(\overline{G}_\sigma)$ is a finite group of Lie type over field of characteristic p and every finite group of Lie type (both split and twisted) has such property. If $\overline{G}$ is simply connected, then $\overline{G}_\sigma = O^{p'}(\overline{G}_\sigma)$ is universal. The rank of $\overline{G}$ is called the *rank* of $O^{p'}(\overline{G}_\sigma)$ Note, that if group $G = O^{p'}(\overline{G}_\sigma)$ is simple, then group $\overline{G}_\sigma$ is the inner-diagonal automorphism group of G. For every σ-stable maximal torus $\overline{T}$ of $\overline{G}$ we have $\overline{T}_\sigma G = \overline{G}_\sigma$.

Suppose $\overline{T}$ to be some σ-stable torus of connected simple algebraic group $\overline{G}$. The group $\overline{T}_\sigma$ (resp. $\overline{T}_\sigma \cap O^{p'}(\overline{G}_\sigma)$) is called a *torus* of $\overline{G}_\sigma$ (resp. $O^{p'}(\overline{G}_\sigma)$). If $\overline{T}$ is maximal, then $\overline{T}_\sigma$ (resp. $\overline{T}_\sigma \cap O^{p'}(\overline{G}_\sigma)$) is called *maximal* torus. One to one correspondence between classes of Weil group and classes of maximal tory in finite split group of Lie type is known. (see [10]). If $T_w(q)$ corresponds to $w \in W$, then $|T_w(q)| = f(q)$, where $f(q)$ is the characteristic polynomial of w. The complete information about classes of Weil groups, characteristic polynomials, and centralizers in Weil group for all simple Lie algebras may be found in [11].

Chapter 2

Abelian subgroups of maximal order in some classes of finite groups

§1 Abelian subgroups of maximal order in symmetric and alternative groups

Theorem 2.1.1. *A large Abelian subgroup in an alternating group A_n is conjugate to one of the following groups:*

1) $\langle(1,2,3),\ldots,(3k-2,3k-1,3k)\rangle$ *if* $n=3k$*;*

2) $\langle(1,2)(3,4),(1,3)(2,4),(5,6,7),\ldots,(3k-1,3k,3k+1)\rangle$ *if* $n=3k+1$*;*

3) $\langle(1,2)(3,4),(1,3)(2,4),(5,6)(7,8),(5,7)(6,8),(9,10,11),\ldots,(3k,\,3k+1,\,3k+2)\rangle$ *if* $n=3k+2$*,* $k\geqslant 2$*;*

4) $\langle(1,2,3,4,5)\rangle$ *if* $n=5$*.*

Also, the orders of large Abelian subgroups in alternating (A_n) and symmetric (S_n) groups are given thus:

$\mathbf{a}(A_{3n})=3^n$; $\mathbf{a}(A_{3n+1})=4\cdot 3^{n-1}$; $\mathbf{a}(A_{3n+2})=16\cdot 3^{n-2}$; $\mathbf{a}(A_5)=5$; $\mathbf{a}(S_{3n})=3^n$; $\mathbf{a}(S_{3n+1})=4\cdot 3^{n-1}$; $\mathbf{a}(S_{3n+2})=2\cdot 3^n$.

For any n*,* $\mathbf{c}_a(A_n)=1$*.*

Proof. We point out the following well-known fact. Let $H\leq S_n$ and assume that H is Abelian and acts transitively on a set $\{1,\ldots,n\}$. Then $|H|=n$.

In fact, consider a stabilizer $St_H(i)$ of some element $i\in\{1,\ldots,n\}$ in the group H. Since H acts transitively, for any $j\in\{1,\ldots,n\}$, there exists a $\tau\in H$ for which $i^\tau=j$. For any $\sigma\in St_H(i)$, therefore, we have

$$j^\sigma=i^{\tau\sigma}=i^{\sigma\tau}=i^\tau=j,$$

that is, if $\sigma\in St_H(i)$ then $\sigma\in St_H(j)$ for all $j\in\{1,\ldots,n\}$. Hence $\sigma=\varepsilon$ is an identical permutation, that is, $St_H(i)=\{\varepsilon\}$. Furthermore, $|H|=|H:St_H(i)|\cdot|St_H(i)|$, $|H:St_H(i)|=n$, and consequently $|H|=n$.

Finally, the whole set $\{1,\ldots,n\}$ splits into disjoint subsets $I_1,\ldots,I_k$, on each of which the Abelian subgroup G of S_n acts transitively. Thus $|G|\leqslant\prod_{j=1}^k|I_j|$.

Write $P_n=\max_{n_1+\ldots+n_k=n}(\prod_{j=1}^k n_j)$. By the above, $\mathbf{a}(S_n)=P_n$. It is not hard to see that P_n satisfies the following recurrent relation:

$$P_n=\max_{0<m\leqslant n}(P_{n-m}\cdot m),\ P_0=1.$$

Using this, by induction we obtain the equalities

$$P_{3n} = 3^n;\ P_{3n+1} = 4 \cdot 3^{n-1};\ P_{3n+2} = 2 \cdot 3^n.$$

The theorem is proved for $\mathbf{A}(S_n)$.

Note that $A_n < S_n$, and hence $\mathbf{a}(A_n) \leqslant \mathbf{a}(S_n)$. In the group A_{3n}, there exists an Abelian subgroup G generated by permutations $(1,2,3), (4,5,6), \dots, (3n-2, 3n-1, 3n)$, that is, G can be represented as a direct product of cyclic groups of order 3. The order of G is equal to 3^n; therefore, $\mathbf{a}(A_{3n}) = 3^n$. It is worth mentioning that any large Abelian subgroup F in A_{3n} is represented as a direct product of cyclic groups of order 3, that is, it is generated by permutations $(k_1, k_2, k_3), (k_4, k_5, k_6), \dots, (k_{3n-2}, k_{3n-1}, k_{3n})$; therefore, $G^\sigma = F$, where σ is a permutation in S_{3n} sending 1 to k_1, 2 to k_2, and so on. If σ is odd, we may take a permutation $(1,2)\sigma = \tau$, which is even. Since $G^{(1,2)} = G$, we have $G^\tau = F$, that is, G and F are conjugate in A_{3n}.

In the group A_{3n+1}, there is an Abelian subgroup G generated by permutations $(1,2)(3,4)$, $(1,3)(2,4)$, $(5,6,7)$, ..., $(3n-1, 3n, 3n+1)$; its order is equal to $4 \cdot 3^{n-1}$, and so $\mathbf{a}(A_{3n}) = 4 \cdot 3^{n-1}$. A proof that any large Abelian group and G are conjugate in A_{3n+1} goes along the same line as in the A_{3n} case.

Lastly, if G is an Abelian subgroup of A_{3n+2}, then either $|G| = 3n+2$, or G is represented as $G = G_1 \times G_2$, where $G_1 < A_{k_1}$, $G_2 < A_{k_2}$, and $k_1 + k_2 = 3n+2$. If $n \geqslant 2$, for the indices k_1 and k_2, we have the following cases.

1. Let $k_1 = 3n_1 + 1$ and $k_2 = 3n_2 + 1$. Then $|G| = |G_1 \times G_2| = |G_1| \cdot |G_2| \leqslant \mathbf{a}(A_{3n_1+1}) \cdot \mathbf{a}(A_{3n_2+1}) = 16 \cdot 3^{n-2}$.

2. Let $k_1 = 3n_1$ and $k_2 = 3n_2 + 2$. Then $|G| = |G_1 \times G_2| = |G_1| \cdot |G_2| \leqslant \mathbf{a}(A_{3n_1}) \cdot \mathbf{a}(A_{3n_2+2}) = 16 \cdot 3^{n-2}$.

Using induction on n yields the estimate $\mathbf{a}(A_{3n+2}) \leqslant 16 \cdot 3^{n-2}$. On the other hand, for $n \geqslant 2$, A_{3n+2} contains a subgroup G generated by permutations $(1,2)(3,4)$, $(1,3)(2,4)$, $(5,6)(7,8)$, $(5,7)(6,8)$, $(9,10,11)$, ..., $(3n, 3n+1, 3n+2)$; its order is equal to $16 \cdot 3^{n-2}$, and so $\mathbf{a}(A_{3n}) = 16 \cdot 3^{n-2}$. As above, we can prove that any large Abelian subgroup and G are conjugate in A_{3n+2}. ⊠

In the table below we assemble information about large abelian subgroups in symmetric and alternative groups.

Table 2.1. Large abelian subgroups of symmetric and alternative groups

G	$\mathbf{a}(G)$	$\mathbf{c}_a(G)$	structure
A_{3n}, S_{3n}	3^n	1	$\langle(1,2,3)\rangle \times \ldots \times \langle(3n-2, 3n-1, 3n)\rangle$
A_{3n+1}	$4 \cdot 3^{n-1}$	1	$\langle(1,2)(3,4),(1,3)(2,4)\rangle \times \langle(5,6,7)\rangle \times \ldots \times$ $\times\langle(3n-1, 3n, 3n+1)\rangle$
S_{3n+1}	$4 \cdot 3^{n-1}$	3	$\langle(1,2)\rangle \times \langle(3,4)\rangle \times \langle(5,6,7)\rangle \ldots \times \langle(3n-1, 3n, 3n+1)\rangle$, $A \in \mathbf{A}(A_{3n+1})$, $\langle(1,2,3,4)\rangle \times \langle(5,6,7)\rangle \times \ldots \times$ $\times\langle(3n-1, 3n, 3n+1)\rangle$
A_{3n+2}, $n \geqslant 2$	$16 \cdot 3^{n-2}$	1	$\langle(1,2)(3,4),(1,3)(2,4)\rangle \times \langle(5,6)(7,8),(5,7)(6,8)\rangle \times$ $\times\langle(9,10,11)\rangle \times \ldots \times \langle(3n, 3n+1, 3n+2)\rangle$
S_{3n+2}	$2 \cdot 3^n$	1	$\langle(1,2)\rangle \times \langle(3,4,5)\rangle \times \ldots \times \langle(3n, 3n+1, 3n+2)\rangle$
A_5	5	1	$\langle(1,2,3,4,5)\rangle$

§2 Abelian unipotent subgroups of finite Chevalley groups. Main theorems

Theorem 2.2.1. *Let G be a finite split Chevalley group with base field $GF(q)$. $L \leq U$ is assumed to be some unipotent subgroup of G. Then $|L| \leqslant q^{|m(L)|}$.*

Proof. For all $r \in m(L)$ consider a maximal subset $X(r) = \{x(r,t_r) | t_r \in GF(q)\}$ of L such that the following conditions hold.

1. For every $x(r,t_r) \in X(r)$ we have $m(x(r,t_r)) = r$.
2. The element $x_r(t_r)$ is in the canonical decomposition of $x(r,t_r)$.
3. For every two elements $x(r,t_r)$ and $x(r,u_r)$ of $X(r)$ we have $t_r \neq u_r$.
4. $e \in X(r)$ for all $r \in m(L)$.

Just defined subset $X(r)$ is some analogue in L of root subgroup X_r in U. In general, $X(r)$ is not uniquely determined. But the cardinality of $X(r)$ is constant (for a given group L) and is not greater then $|GF(q)| = q$. We prove the following statement.

We fix some set $\{X(r) | r \in m(L)\}$. Then every element x of L has the following decomposition $x = \prod_{r \in m(L), r \geqslant m(x)} x(r,t_r)$. $(*)$

Assume, that $(*)$ is false and x is a maximal with respect to $m(x)$ counterexample. Suppose that $r = m(x)$ and the element $x_r(t_r)$ of the root subgroup X_r is in the canonical decomposition of x. By definition, there exists an element $x(r,u)$ of $X(r)$ such that $u = t_r$. Then $x = x_r(t_r) \prod_{s>r} x_s(t_s)$ and $(x(r,t_r))^{-1} = x_r(-t_r) \prod_{s>r} x_s(u_s)$. Thus we have.

$$\begin{aligned}(x(r,t_r))^{-1} \cdot x &= x_r(-t_r) \cdot \prod_{s>r} x_s(u_s) \cdot x_r(t_r) \cdot \prod_{s>r} x_s(t_s) = \\ &= x_r(-t_r) \cdot x_r(t_r) \cdot \prod_{s>r} x_s(u_s) \cdot \left[\prod_{s>r} x_s(u_s), x_r(t_r)\right] \cdot \prod_{s>r} x_s(t_s) = \\ &= \prod_{s>r} x_s(u_s) \cdot \left[\prod_{s>r} x_s(u_s), x_r(t_r)\right] \cdot \prod_{s>r} x_s(t_s).\end{aligned}$$

By lemma 1.3.2, the commutator

$$\left[\prod_{s>r} x_s(u_s), x_r(t_r)\right]$$

is in the subgroup $\langle X_s | s > r\rangle$. The elements $\prod_{s>r} x_s(t_s)$ and $\prod_{s>r} x_s(u_s)$ are in $\langle X_s | s > r\rangle$ also. Therefore the element $x_1 = (x(r,t_r))^{-1} \cdot x$ is in $\langle X_s | s > r\rangle$. In particular $m(x_1) > m(x)$. Since $m(x_1) > m(x)$, and x is a maximal with respect to $m(x)$ counterexample, we have that the element x_1 has the decomposition $x_1 = \prod_{s \in m(L), s \geqslant m(x_1)} x(s,t_s)$. Hence the following equality holds

$$x = x(r,t_r) \prod_{s \in m(L), s \geqslant m(x_1)} x(s,t_s) = \prod_{s \in m(L), s \geqslant m(x)} x(s,t_s),$$

a contradiction.

Thus we prove that $|L| \leqslant \prod_{r \in m(L)} |X(r)| \leqslant q^{|m(L)|}$. ⊠

Theorem 2.2.2. *Let G be a finite split group of Lie type over $GF(q)$ of characteristic p and $U \in \mathrm{Syl}_p(G)$. Assume that $x, y \in U$ and $[x, y] = 1$. Then the set $\{m(x), m(y)\}$ is abelian with respect to p. In particular, for an abelian unipotent subgroup $A \leq U$ we have that the set $m(A)$ is abelian with respect to p.*

PROOF. Suppose that $m(x) = r$, $m(y) = s$. So $x = x_r(t)v_1x_{r+s}(t_1)v_2$ and $y = x_s(u)w_1$ $x_{r+s}(u_1)w_2$, where
$v_1 = \prod_{f\in\Phi,\ r<f<r+s} x_f(t_f)$, $v_2 = \prod_{f\in\Phi, f>r+s} x_f(t_f)$, $w_1 = \prod_{f\in\Phi, s<f<r+s} x_f(u_f)$, $w_2 = \prod_{f\in\Phi, f>r+s} x_f(u_f)$ and $t, t_1, t_f, u, u_1, u_f \in GF(q)$.

Assume, that $r \leqslant s$. The we obtain the following identities.

$$
\begin{aligned}
xy = x_r(t)v_1x_{r+s}(t_1)v_2\cdot x_s(u)w_1x_{r+s}(u_1)w_2 =&\\
= x_r(t)\cdot\left(\prod_{f\in\Phi,r<f<r+s} x_f(t_f)\right)\cdot x_{r+s}(t_1)\cdot\left(\prod_{f\in\Phi,f>r+s} x_f(t_f)\right)\times&\\
\times\, x_s(u)\cdot\left(\prod_{f\in\Phi,s<f<r+s} x_f(u_f)\right)\cdot x_{r+s}(u_1)\left(\prod_{f\in\Phi,f>r+s} x_f(u_f)\right)=&\\
= x_r(t)\cdot\left(\prod_{f\in\Phi,r<f<s} x_f(t_f)\right)\cdot x_s(u)\cdot\left(\prod_{f\in\Phi,s\leqslant f<r+s} x_f(t_f)\right)\cdot x_{r+s}(t_1)\cdot\left(\prod_{f\in\Phi,f>r+s} x_f(t_f)\right)\times&\\
\times\left[\left(\prod_{f\in\Phi,s\leqslant f<r+s} x_f(t_f)\right)\cdot x_{r+s}(t_1)\cdot\left(\prod_{f\in\Phi,f>r+s} x_f(t_f)\right), x_s(u)\right]\times&\\
\times\left(\prod_{f\in\Phi,s<f<r+s} x_f(u_f)\right)\cdot x_{r+s}(u_1)\cdot\left(\prod_{f\in\Phi,f>r+s} x_f(u_f)\right)=&\\
= x_r(t)\cdot\left(\prod_{f\in\Phi,r<f<s} x_f(t_f)\right)\cdot x_s(u+t_s)\cdot\left(\prod_{f\in\Phi,s<f<r+s} x_f(t_f)\right)\cdot x_{r+s}(t_1)\times&\\
\left(\prod_{f\in\Phi,s<f<r+s} x_f(u_f)\right)\cdot x_{r+s}(u_1)\times&\\
\times\left(\prod_{f\in\Phi,f>r+s} x_f(t_f)\right)\cdot\left[\left(\prod_{f\in\Phi,s\leqslant f<r+s} x_f(t_f)\right)\cdot x_{r+s}(t_1)\cdot\left(\prod_{f\in\Phi,f>r+s} x_f(t_f)\right), x_s(u)\right]\times&\\
\times\left[\left(\prod_{f\in\Phi,f>r+s} x_f(t_f)\right)\cdot\left[\left(\prod_{f\in\Phi,s\leqslant f<r+s} x_f(t_f)\right)\cdot x_{r+s}(t_1)\cdot\left(\prod_{f\in\Phi,f>r+s} x_f(t_f)\right), x_s(u)\right],\right.&\\
\left.\left(\prod_{f\in\Phi,s<f<r+s} x_f(u_f)\right)\cdot x_{r+s}(u_1)\right]\cdot\left(\prod_{f\in\Phi,f>r+s} x_f(u_f)\right)=&\\
= x_r(t)\cdot\left(\prod_{f\in\Phi,r<f<s} x_f(t_f)\right)\cdot x_s(u+t_s)\cdot\left(\prod_{f\in\Phi,s<f<r+s} x_f(t_f+u_f)\right)\cdot x_{r+s}(t_1+u_1)\times&\\
\times\, z\cdot\left(\prod_{f\in\Phi,f>r+s} x_f(t_f)\right)\cdot\left[\left(\prod_{f\in\Phi,s\leqslant f<r+s} x_f(t_f)\right)\cdot x_{r+s}(t_1)\cdot\left(\prod_{f\in\Phi,f>r+s} x_f(t_f)\right), x_s(u)\right]\times&\\
\times\left[\left(\prod_{f\in\Phi,f>r+s} x_f(t_f)\right)\cdot\left[\left(\prod_{f\in\Phi,s\leqslant f<r+s} x_f(t_f)\right)\cdot x_{r+s}(t_1)\cdot\left(\prod_{f\in\Phi,f>r+s} x_f(t_f)\right), x_s(u)\right],\right.&\\
\left.\left(\prod_{f\in\Phi,s<f<r+s} x_f(u_f)\right)\cdot x_{r+s}(u_1)\right]\cdot\left(\prod_{f\in\Phi,f>r+s} x_f(u_f)\right),&
\end{aligned}
$$

where the element z is defined by the following way. Consider the minimal root f_1 of

the set $\Psi_1 = \{s < f \leqslant r+s | f \in \Phi\}$. Then the element

$$\left(\prod_{f \in \Phi, s<f<r+s} x_f(t_f) \right) \cdot x_{r+s}(t_1) \cdot x_{f_1}(u_{f_1})$$

is equal to

$$x_{f_1}(u_{f_1}) \cdot \left(\prod_{f \in \Phi, s<f<r+s} x_f(t_f) \right) \cdot x_{r+s}(t_1) \cdot \left[\left(\prod_{f \in \Phi, s<f<r+s} x_f(t_f) \right) \cdot x_{r+s}(t_1), x_{f_1}(u_{f_1}) \right].$$

Put

$$z_1 = \left[\left(\prod_{f \in \Phi, s<f<r+s} x_f(t_f) \right) \cdot x_{r+s}(t_1), x_{f_1}(u_{f_1}) \right].$$

Then we take the minimal root f_2 of $\Psi_2 = \{f_1 < f \leqslant r+s | f \in \Phi\}$. Now the element

$$\left(\prod_{f \in \Phi, f_1<f<r+s} x_f(t_f) \right) \cdot x_{r+s}(t_1) \cdot z_1 \cdot x_{f_2}(u_{f_2})$$

equals

$$x_{f_2}(u_{f_2}) \left(\prod_{f \in \Phi, f_1<f<r+s} x_f(t_f) \right) x_{r+s}(t_1) z_1 \left[\left(\prod_{f \in \Phi, s<f<r+s} x_f(t_f) \right) x_{r+s}(t_1) z_1, x_{f_2}(u_{f_2}) \right].$$

Put

$$z_2 = z_1 \cdot \left[\left(\prod_{f \in \Phi, s<f<r+s} x_f(t_f) \right) \cdot x_{r+s}(t_1) \cdot z_1, x_{f_2}(u_{f_2}) \right].$$

We repeat this procedure until we obtain the empty set Ψ_i. Since the set Ψ_1 is finite and for every k we have $|\Psi_k| > |\Psi_{k+1}|$, this procedure stops after a finite number of steps. Put z to be equal to z_{i-1}. By lemma 1.3.2, the element z is in $\langle X_f | f \geqslant f_1 + f_2 \rangle$. Since $f_1 > r$ and $f_2 > s$ we have $m(z) > r+s$, therefore $z \in \langle X_f | f > r+s \rangle$.

Thus, by lemma 1.3.2 we obtain, that all root factors in the canonical form of the element

$$z \left(\prod_{f \in \Phi, f>r+s} x_f(t_f) \right) \cdot \left[\left(\prod_{f \in \Phi, s \leqslant f<r+s} x_f(t_f) \right) \cdot x_{r+s}(t_1) \cdot \left(\prod_{f \in \Phi, f>r+s} x_f(t_f) \right), x_s(u) \right] \times$$
$$\times \left[\left(\prod_{f \in \Phi, f>r+s} x_f(t_f) \right) \cdot \left[\left(\prod_{f \in \Phi, s \leqslant f<r+s} x_f(t_f) \right) \cdot x_{r+s}(t_1) \cdot \left(\prod_{f \in \Phi, f>r+s} x_f(t_f) \right), x_s(u) \right], \right.$$
$$\left. \left(\prod_{f \in \Phi, s<f<r+s} x_f(u_f) \right) \cdot x_{r+s}(u_1) \right] \cdot \left(\prod_{f \in \Phi, f>r+s} x_f(u_f) \right)$$

are in $\langle X_f | f > r+s \rangle$. Hence, this element is in $\langle X_f | f > r+s \rangle$. So, we may write it in the following way $\prod_{f>r+s} x_f(a_f)$, where a_f are elements of $GF(q)$. Thus we obtain the following canonical form of xy.

$$xy = x_r(t)\cdot\left(\prod_{f\in\Phi, r<f<s} x_f(t_f)\right)\cdot x_s(u+t_s)\cdot\left(\prod_{f\in\Phi, s<f<r+s} x_f(t_f+u_f)\right)\times$$
$$\times\, x_{r+s}(t_1+u_1)\cdot\prod_{f>r+s} x_f(a_f). \quad (2.1)$$

Applying analogous arguments to the element yx we obtain the following canonical form

$$yx = x_r(t)\left(\prod_{f\in\Phi, r<f<s} x_f(t_f)\right) x_s(u+t_s)\left(\prod_{f\in\Phi, s<f<r+s} x_f(t_f+u_f)\right)\times$$
$$\times\, x_{r+s}(t_1+u_1-C_{11rs}tu)\left(\prod_{f\in\Phi, f>r+s} x_f(b_f)\right), \quad (2.2)$$

where all elements b_f are in $GF(q)$. By (2.1) and (2.2), it follows that if the set $\{m(x), m(y)\}$ is not abelian with respect to p, then canonical forms of xy and yx are not coincide, since the element $x_{r+s}(t_1+u_1)$ differs from the element $x_{r+s}(t_1+u_1-C_{11rs}tu)$. By uniqueness of canonical form, we have, that $xy \neq yx$, a contradiction. ⊠

Corollary 2.2.3. *Let G be a finite split group of Lie type over $GF(q)$ of characteristic p. Let A be an abelian unipotent subgroup of G and Φ the root system of G. Then $|A| \leqslant q^{\mathbf{a}(\Phi,p)}$.*

Proof. Since A is unipotent, we may assume, that $A \leq U$. By theorem 2.2.2 the set $m(A)$ is abelian with respect to p. So, by theorem 2.2.1, $|A| \leqslant q^{m(A)} \leqslant q^{\mathbf{a}(\Phi,p)}$. ⊠

Theorem 2.2.4. *Let G be a finite split group of Lie type. Let A_1 and A_2 be two subgroups of U and $B = N_G(U)$ a Borel subgroup of G, A_1 and A_2 are conjugate in B. Then $m(A_1) = m(A_2)$.*

Proof. Suppose, that $A_1^g = A_2$ for some $g \in B$. Then $g = hu$, where h is an element of Cartan subgroup H of B and u is in U. For every root $r \in \Phi^+$ we have $X_r^H \subseteq X_r$, so $X_r^h \subseteq X_r$. By lemma 1.3.2 we have, that for every $u \in U$ the following inclusion holds $X_r^u \subseteq X_r\langle X_s | s > r\rangle$. Assume that $x \in A_1$ and $r = m(x)$. By previous arguments we obtain that $x^g \in X_r\langle X_s | s > r\rangle$. Hence, $m(x^g) = m(x) \in m(A_2)$. Therefore, $m(A_1) \subseteq m(A_2)$. Since the identity $A_2^{g^{-1}} = A_1$ holds, we get $m(A_2) \subseteq m(A_1)$. Thus, $m(A_1) = m(A_2)$. ⊠

§3 Large abelian unipotent subgroups in $G_2(q)$

The structure and orders of large abelian unipotent subgroups and Thompson subgroups in maximal unipotent subgroups in $G_2(q)$ depend on the characteristic p of the base field $GF(q)$ of $G_2(q)$.

The characteristic equals 2 Maximal abelian with respect to 2 subsets of Φ^+ are given in the list below.

$$\{r_1, r_1+r_2, 3r_1+r_2, 3r_1+2r_2\},$$
$$\{r_2, r_1+r_2, 3r_1+2r_2\},$$
$$\{r_2, 2r_1+r_2, 3r_1+2r_2\},$$
$$\{2r_1+r_2, 3r_1+r_2, 3r_1+2r_2\}.$$

Consider the first member of the list. Direct computations show, that for every $x, y \in U$ such that $m(x) = r_1$ and $m(y) = r_1 + r_2$ we have $xy \neq yx$. Thus, if A is an abelian subgroup such that

$$m(A) \subseteq \{r_1, r_1+r_2, 3r_1+r_2, 3r_1+2r_2\},$$

then either

$$m(A) \subseteq \{r_1, 3r_1+r_2, 3r_1+2r_2\},$$

or

$$m(A) \subseteq \{r_1+r_2, 3r_1+r_2, 3r_1+2r_2\}.$$

Consider former case. Up to conjugation A coincide with one of the following groups either

$$\langle X_{r_1}, X_{3r_1+r_2}, X_{3r_1+2r_2}\rangle$$

or

$$\langle \{x_{r_1}(a)x_{r_1+r_2}(a) | a \in GF(q)\}, X_{3r_1+r_2}, X_{3r_1+2r_2}\rangle.$$

Anyway, by theorem 2.2.1, we get $|A| \leqslant q^3$. Analogous arguments give us a complete list of large abelian subgroups of U). We do not give this list here. Note that

$$\langle X_{r_1}, X_{3r_1+r_2}, X_{3r_1+2r_2}\rangle$$

and

$$\langle X_{r_2}, X_{r_1+r_2}, X_{3r_1+2r_2}\rangle$$

are large elementary abelian subgroups of U and generate U.

$$\langle X_{2r_1+r_2}, X_{3r_1+r_2}, X_{3r_1+2r_2}\rangle$$

is a unique large abelian normal subgroup of U. Hence, $\mathbf{a}(U) = \mathbf{a}_e(U) = \mathbf{a}_n(U) = q^3$, $J(U) = J_e(U) = U$,

$$J_n(U) = \langle X_{2r_1+r_2}, X_{3r_1+r_2}, X_{3r_1+2r_2}\rangle,$$

$m_2(G_2(2^\alpha)) = 3\alpha$.

The characteristic equals 3 In this case abelian with respect to 3 subsets of Φ^+ can be found in the list below.

$$\{r_1, 2r_1 + r_2, 3r_1 + r_2, 3r_1 + 2r_2\},$$
$$\{r_1 + r_2, 2r_1 + r_2, 3r_1 + r_2, 3r_1 + 2r_2\},$$
$$\{r_2, r_1 + r_2, 2r_1 + r_2, 3r_1 + 2r_2\}.$$

Thus, large abelian unipotent subgroup A of $U \in \mathrm{Syl}_p(G_2(q))$ is conjugate (in U) to either

$$\langle X_{r_1}, X_{2r_1+r_2}, X_{3r_1+r_2}, X_{3r_1+2r_2}\rangle,$$

or

$$\langle \{x_{r_1}(a)x_{r_1+r_2}(a) | a \in GF(q)\}, X_{2r_1+r_2}, X_{3r_1+r_2}, X_{3r_1+2r_2}\rangle,$$

if $m(A)$ coincide with the first set. A is conjugate to

$$\langle X_{r_1+r_2}, X_{2r_1+r_2}, X_{3r_1+r_2}, X_{3r_1+2r_2}\rangle,$$

if $m(A)$ coincide with the second set. A is conjugate to either

$$\langle X_{r_2}, X_{r_1+r_2}X_{2r_1+r_2}, X_{3r_1+2r_2}\rangle,$$

or

$$\langle \{x_{r_2}(a)x_{3r_1+r_2}(a) | a \in GF(q)\}, X_{r_1+r_2}, X_{2r_1+r_2}, X_{3r_1+2r_2}\rangle,$$

if $m(A)$ coincide with the third set. Thus, $\mathbf{a}(U) = \mathbf{a}_e(U) = \mathbf{a}_n(U) = q^4$, $J(U) = J_e(U) = U$,

$$J_n(U) = \langle X_{r_1+r_2}, X_{2r_1+r_2}, X_{3r_1+r_2}, X_{3r_1+2r_2}\rangle,$$

and $m_3(G_2(3^\alpha)) = 4\alpha$.

The characteristic is greater then 3 In this case all abelian with respect to p subsets of Φ^+ are abelian and may be found below.

$$\{r_1, 3r_1 + r_2, 3r_1 + 2r_2\},$$
$$\{r_2, r_1 + r_2, 3r_1 + 2r_2\},$$
$$\{r_2, 2r_1 + r_2, 3r_1 + 2r_2\},$$
$$\{r_1 + r_2, 3r_1 + r_2, 3r_1 + 2r_2\},$$
$$\{2r_1 + r_2, 3r_1 + r_2, 3r_1 + 2r_2\}.$$

We do not give a complete list of large abelian subgroups in this case. Note, that groups

$$\langle X_{r_1}, X_{3r_1+r_2}, X_{3r_1+2r_2}\rangle$$

and

$$\langle X_{r_2}, X_{r_1+r_2}X_{3r_1+2r_2}\rangle$$

are large elementary abelian, unipotent and generate U.

$$\langle X_{2r_1+r_2}, X_{3r_1+r_2}, X_{3r_1+2r_2}\rangle$$

is a unique normal large abelian subgroup. Hence, $\mathbf{a}(U) = \mathbf{a}_e(U) = \mathbf{a}_n(U) = q^3$, $J(U) = J_e(U) = U$,

$$J_n(U) = \langle X_{2r_1+r_2}, X_{3r_1+r_2}, X_{3r_1+2r_2}\rangle,$$

and $m_p(G_2(p^\alpha)) = 3\alpha$.

§4 Large abelian unipotent subgroups in ${}^3D_4(q^3)$

Since all roots of D_4 have the same length, all constants $|C_{ijrs}|$ equals 1. Therefore, for all primes abelian with respect to p subset of Φ is abelian. In the table 2.2 below we give the list of maximal abelian subsets of D_4^+. In the table the quadruple $\alpha_1\alpha_2\alpha_3\alpha_4$ corresponds to the root $\alpha_1 r_1 + \alpha_2 r_2 + \alpha_3 r_3 + \alpha_4 r_4$.

Table 2.2. Maximal abelian subsets of D_4^+

1000 0010 0001 1111 1211	1000 0010 1110 1111 1211	1000 0001 1101 1111 1211	1000 1100 1110 1101 1111 1211	0100 1100 0110 0101 1211	0100 1100 0110 1110 1211	0100 1100 0101 1101 1211	0100 1100 1110 1101 1211	0100 0110 0101 0111 1211	0100 0110 1110 0111 1211
0100 1110 1101 0111 1211	0010 0001 0111 1111 1211	0010 0110 1110 0111 1111 1211	0001 0101 1101 0111 1111 1211	1100 0110 0101 1111 1211	1100 0110 1110 1111 1211	1100 0101 1101 1111 1211	0110 0101 0111 1111 1211	1110 1101 0111 1111 1211	0100 0101 1101 0111 1211

Assume that A is an abelian subgroup of ${}^3D_4(q^3)$. Then A is an abelian subgroup of $D_4(q^3)$ and A consists of σ-stable points, where σ is the graph automorphism of order 3. By theorem 2.2.2, the set $m(A)$ is a subset of some set from table 2.2.

Suppose that $m(A) \subseteq \{r_1, r_3, r_4, r_1 + r_2 + r_3 + r_4, r_1 + 2r_2 + r_3 + r_4\}$. Define sets $X(r)$ and elements $x(r,t)$ as in the proof of theorem 2.2.1. Then, by proof of theorem 2.2.1, we have that $|A| \leqslant \prod_{r\in m(A)} |X(r)|$. Assume that $r_1 \in m(A)$. Then there exists an element $x \in A$ such that $m(x) = r_1$. Since x is σ-stable, factors $x_{r_3}(t_{r_3})$ and $x_{r_4}(t_{r_4})$ of the canonical form of x are different from the unit. Conversely, if $x_{r_3}(t_{r_3})$ or $x_{r_4}(t_{r_4})$ is different from the unit, then $x_{r_1}(t_{r_1}) \neq 1$. Thus $r_3, r_4 \notin m(A)$. Then note, that $|X(r_1)| \leqslant q^3$. Roots $r_1 + r_2 + r_3 + r_4$ and $r_1 + 2r_2 + r_3 + r_4$ are ρ-stable, hence for any $x(r,t)$ ($r = r_1 + r_2 + r_3 + r_4$ or $r = r_1 + 2r_2 + r_3 + r_4$) we have $t = \bar{t} = \bar{\bar{t}} \in GF(q)$. Consequently, $|X(r)| \leqslant q$ (if $r = r_1 + r_2 + r_3 + r_4$ or $r = r_1 + 2r_2 + r_3 + r_4$), therefore $|A| \leqslant q^5$. From the other hand,

$$\langle \{x_{r_1}(t), x_{r_3}(\bar{t}), x_{r_4}(\bar{\bar{t}}) | t \in GF(q^3)\}, \{x_{r_1+r_2+r_3+r_4}(t), x_{r_1+2r_2+r_3+r_4}(s) | t, s \in GF(q)\}\rangle$$

is an abelian subgroup of order q^5.

Suppose, that $m(A) \subseteq \{r_1, r_3, r_1 + r_2 + r_3, r_1 + r_2 + r_3 + r_4, r_1 + 2r_2 + r_3 + r_4\}$. Then, by calculation with Chevalley commutator formulae (lemma 1.3.2), we obtain, that if $m(x) = r_1$ and $m(y) = r_1 + r_2 + r_3$, then $xy \neq yx$. Hence, either r_1, or $r_1 + r_2 + r_3$ do not belongs to $m(A)$. As before, we have, that $|A| \leqslant q^5$.

In other cases we produce in the same way.

We do not give the list of large abelian unipotent subgroups of ${}^3D_4(q^3)$. Note that

$$\langle \{x_{r_1}(t), x_{r_3}(\bar{t}), x_{r_4}(\bar{\bar{t}}) | t \in GF(q^3)\}, \{x_{r_1+r_2+r_3+r_4}(t), x_{r_1+2r_2+r_3+r_4}(s) | t, s \in GF(q)\}\rangle$$

and

$$\langle\{x_{r_1+r_2}(t), x_{r_2+r_3}(\bar{t}), x_{r_2+r_4}(\bar{\bar{t}}) | t \in GF(q^3)\}, \{x_{r_2}(t), x_{r_1+2r_2+r_3+r_4}(s) | t, s \in GF(q)\}\rangle$$

are large elementary abelian and generate U. The unique large normal abelian unipotent subgroups is conjugate to A, where

$$\begin{aligned} A = \langle\{&x_{r_1+r_2+r_3}(t)x_{r_1+r_2+r_4}(\bar{t})x_{r_2+r_3+r_4}(\bar{\bar{t}}) | t \in GF(q^3)\}, \\ &\{x_{r_1+r_2+r_3+r_4}(t), x_{r_1+2r_2+r_3+r_4}(s) | t, s \in GF(q)\}\rangle. \end{aligned}$$

Hence, $\mathbf{a}(U) = \mathbf{a}_e(U) = \mathbf{a}_n(U) = q^5$, $J(U) = J_e(U) = U$, $J_n(U) = A$, and $m_p({}^3D_4(p^{3\alpha})) = 5\alpha$.

§5 Large abelian unipotent subgroups of $F_4(q)$ and ${}^2F_4(q)$

The structure and orders of large abelian unipotent subgroups and Thompson subgroups in maximal unipotent subgroups of $F_4(q)$ depend on q, so we consider two cases.

q **is even** In the table 2.3 we give the list of all maximal abelian with respect to 2 subsets of F_4^+. As in table 2.2, the quadruple $\alpha_1\alpha_2\alpha_3\alpha_4$ corresponds to the root $\alpha_1 r_1 + \alpha_2 r_2 + \alpha_3 r_3 + \alpha_4 r_4$.

Table 2.3. Abelian with respect to 2 subsets of F_4^+

1000	1000	1000	1000	1000	1000	1000	1000	0100	0100	0100	0120	0111
0010	0010	0010	0001	0001	0001	0011	0011	0001	0001	0001	0121	1111
0011	1110	1110	1100	0011	1111	1120	1111	1100	0111	0111	1121	1220
1120	1120	1120	1111	1111	1121	1111	1121	0111	1111	0121	0122	1221
1121	1220	1121	1121	1121	1221	1121	1122	1111	0122	0122	1221	1231
1122	1121	1122	1221	1122	1122	1122	1231	1221	1221	0122	1231	1222
1231	1231	1231	1122	1222	1222	1231	1222	1222	1222	1222	1232	1232
1232	1232	1232	1222	1232	1232	1232	1232	1232	1232	1232	1242	1242
1242	1242	1242	1232	1242	1242	1242	1242	1342	1342	1342	1342	1342
2342	2342	2342	2342	2342	2342	2342	2342	2342	2342	2342	2342	2342
1000	1000	1000	1000	1000	1000	1000	1000	0100	0100	0100	0121	1111
1100	1100	1100	1100	1110	1110	1110	1110	1100	1100	1100	1121	1121
1110	1110	1110	1110	1120	1120	1111	1111	0110	0110	1110	0122	0122
1120	1120	1111	1111	1111	1111	1220	1121	1110	0111	1111	1221	1221
1111	1111	1220	1121	1220	1121	1121	1221	1220	1220	1220	1122	1122
1220	1121	1121	1221	1121	1221	1221	1122	1221	1221	1221	1231	1231
1121	1221	1221	1122	1221	1122	1231	1231	1231	1231	1231	1222	1222
1221	1122	1231	1231	1231	1231	1222	1222	1222	1222	1222	1232	1232
1231	1231	1222	1222	1232	1232	1232	1232	1232	1232	1232	1242	1242
1232	1232	1232	1232	1242	1242	1242	1242	1342	1342	1342	1342	1342
2342	2342	2342	2342	2342	2342	2342	2342	2342	2342	2342	2342	2342

0100 1100 0111 1111 1220 1221 1231 1222 1232 1342 2342	0100 0110 0120 0111 0121 1220 1221 1231 1232 1342 2342	0100 0110 0120 0111 0121 0122 1221 1231 1232 1342 2342	0100 0110 0111 0121 1220 1221 1231 1222 1232 1342 2342	0100 0110 0111 0121 0122 1221 1231 1222 1232 1342 2342	0010 0110 1110 0120 1120 1220 1231 1232 1242 1342 2342	0010 0110 0120 1120 0121 1220 1231 1232 1242 1342 2342	0010 0011 0120 1120 0121 1121 1231 1232 1242 1342 2342	0010 0011 0120 0121 1121 0122 1231 1232 1242 1342 2342	0010 0011 1120 0121 1121 1122 1231 1232 1242 1342 2342	0010 0011 0121 1121 0122 1122 1231 1232 1242 1342 2342	0111 1111 0122 1221 1122 1231 1222 1232 1242 1342 2342	0111 0121 0122 1221 1122 1231 1222 1232 1242 1342 2342
0100 0110 1110 0120 1220 1221 1231 1232 1342 2342	0100 0111 1111 0122 1221 1231 1222 1232 1342 2342	0010 0110 0120 0121 0122 1231 1232 1242 1342 2342	0010 1110 1120 1121 1122 1231 1232 1242 1342 2342	0001 1100 0111 1111 1221 1122 1222 1232 1342 2342	0001 1100 1111 1121 1221 1122 1222 1232 1342 2342	1100 0110 1110 1120 1220 1221 1231 1232 1342 2342	1100 0111 1111 1221 1122 1231 1222 1232 1342 2342	0110 1110 1220 1221 1231 1222 1232 1242 1342 2342	0011 0120 0111 0121 0122 1231 1232 1242 1342 2342	0011 1120 1111 1121 1122 1231 1232 1242 1342 2342	1120 0121 1121 1121 1122 1122 1232 1242 1342 2342	0121 1220 1121 1121 1231 1231 1232 1242 1342 2342
0010 1110 0120 1120 1220 1121 1231 1232 1242 1342 2342	0010 0120 1120 0121 1220 1121 1231 1232 1242 1342 2342	0001 0011 0111 1111 0122 1122 1222 1232 1242 1342 2342	0001 0011 0111 0121 0122 1122 1222 1232 1242 1342 2342	0001 0011 1111 1121 0122 1122 1222 1232 1242 1342 2342	0001 0011 0121 1121 0122 1122 1222 1232 1242 1342 2342	0001 0111 1111 0122 1221 1122 1222 1232 1242 1342 2342	0001 0111 0121 0122 1221 1122 1222 1232 1242 1342 2342	0001 1111 1121 0122 1221 1122 1222 1232 1242 1342 2342	0001 0121 1121 0122 1221 1122 1222 1232 1242 1342 2342	1100 1110 1120 1111 1220 1121 1221 1231 1232 1342 2342	1110 1111 1121 1221 1122 1231 1222 1232 1242 1342 2342	0120 1120 0121 1220 1121 1221 1231 1232 1242 1342 2342
1100 1110 1120 1111 1121 1221 1122 1231 1232 1342 2342	1100 1110 1111 1220 1121 1221 1231 1222 1232 1342 2342	1100 1110 1111 1121 1221 1122 1231 1222 1232 1342 2342	0110 1110 0120 1120 1220 1221 1231 1232 1242 1342 2342	0110 0120 0111 0121 1220 1221 1231 1232 1242 1342 2342	0110 0120 0111 0121 0122 1221 1231 1232 1242 1342 2342	0110 0120 1120 0121 1220 1221 1231 1232 1242 1342 2342	0110 0111 0121 1220 1221 1231 1222 1232 1242 1342 2342	0110 0111 0121 0122 1221 1231 1222 1232 1242 1342 2342	0011 0111 1111 0122 1122 1231 1222 1232 1242 1342 2342	0011 0111 0121 0122 1122 1231 1222 1232 1242 1342 2342	1110 1120 1111 1121 1221 1122 1231 1232 1242 1342 2342	1110 1111 1220 1121 1221 1231 1222 1232 1242 1342 2342
0011 1111 1121 0122 1122 1231	0011 0121 1121 0122 1122 1231	1110 0120 1120 1220 1121 1221	1110 1120 1111 1220 1121 1221									

1222	1222	1231	1231									
1232	1232	1232	1232									
1242	1242	1242	1242									
1342	1342	1342	1342									
2342	2342	2342	2342									

Evidently, that in $U \in \mathrm{Syl}_2(F_4(q))$ there exists many non conjugating large abelian subgroups (for every abelian with respect to 2 subsets of maximal order we may correspond at least one subgroup). So we do not give the list of all abelian subgroup in this case. Denote by $\Psi_{i,j}$ the maximal abelian with respect to 2 subset such that this subset is in the intersection of the i-th row and j-th column of table 2.3. Then $\langle X_r | r \in \Psi_{2,1}\rangle$, $\langle X_r | r \in \Psi_{2,9}\rangle$, $\langle X_r | r \in \Psi_{3,7}\rangle$, and $\langle X_r | r \in \Psi_{5,3}\rangle$ are large elementary abelian and generates U. In U there exist two large abelian normal subgroups: $A_1 = \langle X_r | r \in \Psi_{2,12}\rangle$, and $A_2 = \langle X_r | r \in \Psi_{2,13}\rangle$. Therefore, $\mathbf{a}(U) = \mathbf{a}_e(U) = \mathbf{a}_n(U) = q^{11}$, $J(U) = J_e(U) = U$, $J_n(U) = \langle A_1, A_2\rangle$, and $m_2(F_4(2^\alpha)) = 11\alpha$.

Consider ${}^2F_4(q)$ and prove, that $\mathbf{a}({}^2F_4(q)) = 2q^5$. Let A be an abelian unipotent subgroup of ${}^2F_4(q)$. Then A is abelian unipotent subgroup of $F_4(q)$, consisting of σ-stable elements, where σ is the graph automorphism. By theorem 2.2.2, $m(A)$ is abelian with respect to 2. So it is contained in some set from table 2.3. Suppose, that $m(A) \subseteq \Psi_{1,1}$.

Let $r_1 \in m(A)$. Then, since A consists of σ-stable elements, for every element $x(r_1, t)$ the factor $x_{r_4}(t)$ of the canonical form of $x(r_1, t)$ is different from the unit. For the same reason, if $r_3 \in m(A)$, then for every element $x(r_3, t)$ the factor $x_{r_2}(t_{r_2}) \neq 1$. But $r_2 < r_3$, hence, $r_2 \in m(A)$, a contradiction. Therefore, $r_3 \notin m(A)$. Similarly, roots $r_3 + r_4$, $r_1 + r_2 + 2r_3 + 2r_4$, $r_1 + 2r_2 + 4r_3 + 2r_4$, and $2r_1 + 3r_2 + 4r_3 + 2r_4$ are also not in $m(A)$. Furthermore, using calculation by Chevalley commutator formulae (lemma 1.3.2), we obtain, that if $m(x) = r_1$ and $m(y) = r_1 + r_2 + 2r_3$, then $xy \neq yx$. Hence either r_1 or $r_1 + r_2 + 2r_3$ is not in $m(A)$. So, $|m(A)| \leqslant 4$. By theorem 2.2.1, $|A| \leqslant q^4$. In other cases we proceed in the same way.

By $X_{\{r\}}$ we denote the group $\langle X_r, X_{\overline{r}}\rangle_\sigma$. If $r + \overline{r} \notin F_4^+$, then $X_{\{r\}}$ is an abelian group of order q. If $r + \overline{r} \in F_4^+$, then $X_{\{r\}}$ is isomorphic to the maximal unipotent subgroup of ${}^2B_2(q)$ and contains $q - 1$ large abelian subgroups of order $2q$. Then every large abelian unipotent subgroup of $U \in \mathrm{Syl}_2({}^2F_4(q))$ is conjugate to either

$$\langle X_{\{r_2+r_3\}}, A_1, X_{\{r_1+2r_2+2r_3\}}, X_{\{r_1+2r_2+2r_3+r_4\}}, X_{\{r_1+2r_2+3r_3+2r_4\}}\rangle$$

or

$$\langle A_2, X_{\{r_1+r_2+r_3+r_4\}}, X_{\{r_1+r_2+2r_3+r_4\}}, X_{\{r_1+2r_2+2r_3+r_4\}}, X_{\{r_1+2r_2+3r_3+r_4\}}\rangle,$$

where $A_1 \in \mathbf{A}(X_{\{r_1+r_2+2r_3\}})$, $A_2 \in \mathbf{A}(X_{\{r_1+r_2+r_3\}})$. This subgroups are obviously normal in U. So $\mathbf{a}(U) = \mathbf{a}_n(U) = 2q^5$, $\mathbf{a}_e(U) = q^5$,

$$J(U) = J_n(U) = \langle X_{\{r\}} | r \succeq r_2 + r_3\rangle,$$

$$J_e(q) = \langle X_{\{r\}} | r \succeq r_2 + r_3, r \neq r_1 + r_2 + r_3, r \neq r_1 + r_2 + 2r_3\rangle,$$

and $m_2({}^2F_4(2^\alpha)) = 5\alpha$.

q **is odd** In this case there exist 25 abelian subsets of F_4^+ of order 9 (see [32]). The unique normal abelian subgroup of maximal order is

$$A_1 = \langle X_{r_2+2r_3+2r_4}, X_{r_1+2r_2+2r_3+r_4}, X_{r_1+r_2+2r_3+2r_4}, X_{r_1+2r_2+3r_3+r_4},$$

$$X_{r_1+2r_2+2r_3+2r_4}, X_{r_1+2r_2+3r_3+2r_4}, X_{r_1+2r_2+4r_3+2r_4}, X_{r_1+3r_2+4r_3+2r_4}, X_{2r_1+3r_2+4r_3+2r_4}\rangle.$$

Thus, $\mathbf{a}(U) = \mathbf{a}_e(U) = \mathbf{a}_n(U) = q^9$, $J(U) = J_e(U) = \langle X_r | r \neq r_1\rangle$, $J_n(U) = A$, and $m_p(F_4(p^\alpha)) = 9\alpha$.

§6 Large abelian unipotent subgroups in $E_6(q)$ and ${}^2E_6(q^2)$

Since in E_6 all roots have the same length, all constants $|C_{ijrs}|$ are equal to 1. Hence, for every prime p abelian with respect to p subset of E_6 is abelian. There exist two abelian subsets of E_6^+ of order 16 (see [32]). So there exist two large abelian subgroups $\langle X_r | r \succeq r_1\rangle$ and $\langle X_r | r \succeq r_6\rangle$ and this subgroups are normal elementary abelian. Therefore, $J(U) = J_e(U) = J_n(U) = \langle X_r | r \succeq r_1 \textit{ or } r \succeq r_6\rangle$, $\mathbf{a}(U) = \mathbf{a}_e(U) = \mathbf{a}_n(U) = q^{16}$, and $m_p(E_6(p^\alpha)) = 16\alpha$.

Abelian subgroups in ${}^2E_6(q^2)$ are studied like abelian subgroups in ${}^3D_4(q^3)$. There exist about 2000 different maximal abelian subsets of E_6^+. We write down only final result. We have $\mathbf{a}(U) = \mathbf{a}_e(U) = \mathbf{a}_n(U) = q^{12}$,

$$J(U) = J_e(U) = \langle X_{\{r\}} | r \succeq r_3 \textit{ or } r \succeq r_4\rangle,$$

$$J_n(U) = A = \langle X_{\{r\}} | r \in \Psi\rangle,$$

where

$$X_{\{r\}} = (X_r X_{\bar{r}})_\sigma,$$

and $m_p({}^2E_6(p^{2\alpha})) = 12\alpha$. Here $\Psi = \{r_1+r_3+r_4+r_5+r_6, r_1+r_2+r_3+2r_4+r_5, r_1+r_2+r_3+r_4+r_5+r_6, r_2+r_3+2r_4+r_5+r_6, r_1+r_2+2r_3+2r_4+r_5, r_1+r_2+r_3+2r_4+r_5+r_6, r_2+r_3+2r_4+2r_5+r_6, r_1+r_2+2r_3+2r_4+r_5+r_6, r_1+r_2+r_3+2r_4+2r_5+r_6, r_1+r_2+2r_3+2r_4+2r_5+r_6, r_1+r_2+2r_3+3r_4+2r_5+r_6, r_1+2r_2+2r_3+3r_4+2r_5+r_6\}$.

§7 Large abelian unipotent subgroups in $E_7(q)$

For every prime p every abelian with respect to p subset of E_7 is again abelian. There exists the unique abelian subset of order 27 (see [32]). So there exist the unique normal elementary abelian subgroup A of order q^{27}. Thus, $J(U) = J_e(U) = J_n(U) = A = \langle X_r | r \succeq r_7\rangle$, $\mathbf{a}(U) = \mathbf{a}_e(U) = \mathbf{a}_n(U) = q^{27}$, and $m_p(E_7(p^\alpha)) = 27\alpha$.

§8 Large abelian unipotent subgroups in $E_8(q)$

Abelian with respect to p subset of E_8 is again abelian for all primes p. Abelian subsets of maximal order are found in [32]. We do not give the list of all large abelian subsets of E_8^+. Note, that $\mathbf{a}(U) = \mathbf{a}_e(U) = \mathbf{a}_n(U) = q^{36}$, $J(U) = J_e(U) = \langle X_r | r \succeq r_1, r \succeq r_3, r \succeq r_4, r \succeq r_5 \;\; r \succeq r_6\rangle$, and $m_p(E_8(p^\alpha)) = 36\alpha$. For brevity we do not give the structure of $J_n(U)$.

§9 Large abelian unipotent subgroups of finite Chevalley groups. Summary table.

Here we give orders of large abelian unipotent subgroups in finite exceptional groups of Lie type (except ${}^2B_2(q)$ and ${}^3G_2(q)$), and the structure of Thompson subgroups. In the table below we assume that $q = p^\alpha$.

Table 2.4. Orders of large abelian unipotent subgroups, p-ranks and Thompson subgroups of maximal unipotent subgroups of finite exceptional groups of Lie type

Group G	$a(U)$	$a_e(U)$	$a_n(U)$	$J(U)$	$m_p(G)$
$G_2(q)$, $p \neq 3$	q^3	q^3	q^3	U	3α
$G_2(q)$, $p = 3$	q^4	q^4	q^4	U	4α
${}^3D_4(q^3)$	q^5	q^5	q^5	U	5α
$F_4(q)$, q is odd	q^9	q^9	q^9	$\langle X_r \| r \neq r_1 \rangle$	9α
$F_4(q)$, q is even	q^{11}	q^{11}	q^{11}	U	11α
${}^2F_4(q)$	$2q^5$	q^5	$2q^5$	$\langle X_{\{r\}} \| r \succeq r_2 + r_3 \rangle$	5α
$E_6(q)$	q^{16}	q^{16}	q^{16}	$\langle X_r \| r \succeq r_1$ or $r \succeq r_6 \rangle$	16α
${}^2E_6(q^2)$	q^{12}	q^{12}	q^{12}	$\langle X_{\{r\}} \| r \succeq r_3$ or $r \succeq r_4 \rangle$	12α
$E_7(q)$	q^{27}	q^{27}	q^{27}	$\langle X_r \| r \succeq r_7 \rangle$	27α
$E_8(q)$	q^{36}	q^{36}	q^{36}	$\langle X_r \| r \succeq r_1, r \succeq r_3, r \succeq r_4, r \succeq r_5$ or $r \succeq r_6 \rangle$	36α

§10 Large abelian subgroups of sporadic groups

In this section we follow the notations used in [15].

If A is an abelian subgroup of some sporadic groups G, then A is contained in G. The structure of maximal subgroups in all sporadic groups, except Fi_{22}, Th, Fi_{23}, J_4, Fi_{24}', B, and M, may be found in [15]. We may obtain bounds of orders of abelian subgroups in sporadic groups by examine maximal subgroups, mentioned above. Maximal subgroups of Fi_{22} are found in [25] and [26]. Maximal subgroups of Th are described by Linton in [29] and [30]. Kleidman, Parker, and Wilson find maximal subgroups of Fi_{23} in [24]. Maximal subgroups of J_4 are found in [27]. Groups Fi_{24} and Fi_{24}' are described by Linton and Wilson in [31]. The structure of maximal subgroups of B may be found in the recent paper [43]. We will consider M by using another technique.

Consider M_{11}. If H is a maximal subgroup of M_{11}, then, up to conjugation, H coincides with one of the following groups: $M_{10} \cong (A_6){}^{\cdot}2$, $L_2(11)$, $(3^2 : Q_8).2$, S_5, or $2 : S_4$. Assume, that some element of $\mathbf{A}(M_{11})$ is contained in subgroup $M_{10} \cong (A_6){}^{\cdot}2$. Then all abelian subgroups of A_6 have orders 2, 3, 4, 5, or 9. If A is abelian subgroup of $(A_6){}^{\cdot}2$, then

there exists subgroups A_1 of A of index at most 2 such that $A_1 \leq A_6$. If A_1 is a 2-group, then $|A_1| \leqslant 4$ and, hence, $|A| \leqslant 8$. If A_1 is a 3-group and $A_1 \neq A$, then A contains an element of order 6. So, A lies in the centralizer of some element of order 6. There exist (up to conjugation) one element of order 6 and the order of its centralizer equals 6 also. So, $|A| = 6$. If A_1 is a 3-group, and $A_1 = A$, then the order of A is less then, or equal to 9. If, at least, the order of A_1 equals 5 and $A_1 \neq A$, then A contains an element of order 10. But M_{11} does not contain any element of order 10, hence $A_1 = A$, and $|A| = 5$. Therefore, $\mathbf{a}(M_{10}) = 9$.

Suppose, that element of $\mathbf{A}(M_{11})$ lies in $L_2(11)$. Then, by table 3.2, $\mathbf{a}(L_2(11)) = 11$, and, moreover, this element is unique up to conjugation. Hence, if an element of $\mathbf{A}(M_{11})$ is contained in $L_2(11)$, then $\mathbf{c}_a(M_{11}) = 1$.

Suppose that an element of $\mathbf{A}(M_{11})$ lies in $M_9 : 2 \cong (3^2 : Q_8).2$. Orders of abelian 2-subgroups and 3-subgroups of $(3^2 : Q_8).2$ are less then or equal to 8 and 9 respectively. If A contains an element of order 2 and an element of order 3, then it contains an element of order 6. Hence, $|A| = 6$ in this case. So, $\mathbf{a}(M_9 : 2) = 9$.

If an element of $\mathbf{A}(M_{11})$ lies in S_5, then, by table , , 2.1, we have that $\mathbf{a}(S_5) = 6$. Assume at least, that an element of $\mathbf{A}(M_{11})$ is in $M_8 : S_3 \cong 2{\cdot}S_4$. If $A \in \mathbf{A}(2{\cdot}S_4)$ is a 2-group, then its order is less than or equal to 8. If $A \in \mathbf{A}(2{\cdot}S_4)$ is a 3-group, then its order is less then or equal to 3. If there exists an element of order 2 and an element of order 3 in $A \in \mathbf{A}(2{\cdot}S_4)$, then $|A| = 6$. Thus, $\mathbf{a}(M_{11}) = \mathbf{a}(L_2(11))$ and $\mathbf{c}_a(M_{11}) = 1$.

In other sporadic groups we proceed in the same way. All results are assembled in the table below. Note, that we do not obtain orders in all sporadic groups. Moreover, even if we obtain $\mathbf{a}(G)$ for some sporadic group G, we may not find $\mathbf{c}_a(G)$.

Consider the group M. Let A be an abelian subgroup of M. There exists a subgroup $(2^{1+24}_{+}){\cdot}Co_1$ of M, containing Sylow 2-subgroup of M. The order of an abelian subgroup of 2^{1+24}_{+} is less then or equal to 2^{13}. By table 2.5, we have that the order of an abelian 2-subgroup of Co_1 is less then or equal to 2^{11}. Therefore, the order of an abelian 2-subgroup of M is less then 2^{24}. There exists a subgroup $(3^{1+12}_{+}){\cdot}2Suz : 2$ of M, containing some Sylow 3-subgroup of M. The order of abelian 3-subgroup of 3^{1+12}_{+} is less then 3^7, by the table 2.5 we have, that the order of an abelian 3-subgroup of Suz is less then or equal to 3^5. Hence, the order of an abelian 3-subgroup of M is less then or equal to 3^{12}. There also exists a subgroup $5^{1+6} : 4J_2 2$ of M, containing a Sylow 5-subgroup of M. The order of an abelian subgroup of 5^{1+6} is less then or equal to 5^4. The order of an abelian 5-subgroup of J_2 is less then or equal to 5^2. So the order of an abelian 5-subgroup of of M is less then or equal to 5^6. For other primes p we have, that for every $S \in \mathrm{Syl}_p(M)$ the following inequality holds $|S| < 2^{24}$. So, if an element of $\mathbf{A}(M)$ is an abelian p-subgroup and $p \geqslant 7$, then $\mathbf{a}(M) \leqslant 2^{24}$.

Assume, that the order of A is divisible by 2 and 3. Then A lies in the in the centralizer of some element of order 3. By [15] we have that A is contained in one of the following group $3{\cdot}Fi_{24}$, $(3^{1+12}_{+}){\cdot}2Suz : 2$, or $S_3 \times Th$. It is easy to see, that $|A| < 2^{24}$. If the order of A is divisible by 2 and 5, then A is contained in the centralizer of an element of order 10. By [15] it follows, that A lies in one of the following group: either $(D_{10} \times HN){\cdot}2$, or $5^{1+6}_{-} : 4J_2 2$. By using table 2.5, one may verify, that $|A| < 2^{24}$. If the order of A is divisible by $p_1 < p_2$ and $p_2 > 5$, then A contains an element of order $p_1 \cdot p_2$, hence A is contained in the centralizer of some element of order $p_1 \cdot p_2$. By [15] we have, that $|A| < 2^{24}$. If the order of A is divisible by 3 and 5, then A contains an element of order 15. Therefore, A lies in the centralizer of some element of order 15. By [15] we have

that $|A| \leqslant 2^{24}$. Thus, $\mathbf{a}(M) \leqslant 2^{24}$. From the other hand, there exists a subgroup 2^{10+16} of M, hence, there exists an abelian subgroup of M of order 2^{18}. Therefore, we prove, that $2^{18} \leqslant \mathbf{a}(M) \leqslant 2^{24}$.

Table 2.5.

G	$\mathbf{a}(G)$		G	$\mathbf{a}(G)$	
M_{11}	$\mathbf{a}(M_{11}) = 11$	11	$O'N$	$\mathbf{a}(O'N) = 81$	3^4
M_{12}	$\mathbf{a}(M_{12}) = 16$	4^2	Co_3	$\mathbf{a}(Co_3) = 3^5$	—
J_1	$\mathbf{a}(J_1) = 19$	19	Co_2	$2^{10} \leqslant \mathbf{a}(Co_2) \leqslant 2^{11}$	—
M_{22}	$\mathbf{a}(M_{22}) = 16$	$2^4, ?$	Fi_{22}	$\mathbf{a}(Fi_{22}) = 2^{10}$	—
J_2	$\mathbf{a}(J_2) = 25$	5^2	HN	$125 \leqslant \mathbf{a}(HN) \leqslant 512$	—
M_{23}	$\mathbf{a}(M_{23}) = 23$	23	Ly	$\mathbf{a}(Ly) = 3^5$	—
HS	$\mathbf{a}(HS) = 64$	4^3	Th	$\mathbf{a}(Th) = 3^6$	—
J_3	$\mathbf{a}(J_3) = 27$	$3^3 \quad 9 \times 3$	Fi_{23}	$\mathbf{a}(Fi_{23}) = 2^{11}$	—
M_{24}	$\mathbf{a}(M_{24}) = 64$	—	Co_1	$\mathbf{a}(Co_1) = 2^{11}$	—
M^cL	$\mathbf{a}(M^cL) = 81$	3^4	J_4	$\mathbf{a}(J_4) = 2^{11}$	—
He	$\mathbf{a}(He) = 64$	—	Fi'_{24}	$3^7 \leqslant \mathbf{a}(Fi'_{24}) \leqslant 3^9$	—
Ru	$2^7 \leqslant \mathbf{a}(Ru) \leqslant 2^9$	—	B	$2^{17} \leqslant \mathbf{a}(B) \leqslant 2^{20}$	—
Suz	$\mathbf{a}(Suz) = 3^5$	—	M	$2^{18} \leqslant \mathbf{a}(M) \leqslant 2^{24}$	—

Chapter 3

Large abelian subgroups of finite Chevalley groups

§1 Intermediary results

Lemma 3.1.1. *Let $R = S * M$ be a σ-stable connected reductive subgroup of maximal rank of connected simple algebraic group G, where $S = (R)^0$, $M = [R, R]$. Let $G_1, \ldots, G_k$ be all simple components of M_σ. Put $z_{i,j} = |Z(G_i) \cap Z(G_j)|$, $z_i = |Z(G_i) \cap S_\sigma|$. Then $O^{p'}(R_\sigma) = G_1 * \ldots * G_k$, $R_1 = G_1 * \ldots * G_k * S_\sigma$ is a normal subgroup of R_σ, and $|R_\sigma : R_1| \leqslant \prod_{i \neq j} z_{i,j} \prod_i z_i$.*

Proof. The identity $O^{p'}(R_\sigma) = G_1 * \ldots * G_k * (S_\sigma)$ and the fact, that $R_1 \trianglelefteq R$ are known. So we prove remaining inequality. By [16, proposition 2.4.2] we have, that $|R_\sigma| = |M_\sigma| \cdot |S_\sigma| = |G_1| \cdot \ldots \cdot |G_k| \cdot |S_\sigma|$. Since $G_i \cap G_j = Z(G_i) \cap Z(G_j)$ for $i \neq j$, we obtain, that $|G_1 * \ldots * G_k * (S_\sigma)| \leqslant |R_\sigma| / (\prod_{i \neq j} z_{i,j} \prod_i z_i)$, and the lemma follows. ⊠

Lemma 3.1.2. *Let G be a connected reductive linear algebraic group defined over algebraically closed field of characteristic p, R its reductive (but not necessary connected) subgroup of maximal rank such that $(|R : R^0|, p) = 1$, $s \in R^0$ is semisimple, and $T$is some maximal torus of R^0, containing s. Then $C_R(s)$ is reductive (but not necessary connected). It is generated by T, root subgroups U_α such that $\alpha(s) = 1$, and elements $n_w \in N_R(T)$, commuting with s. The connected component $C_R(s)^0$ is generated by T and U_α. In particular, $C_R(s)/C_R(s)^0$ is isomorphic to some section of $W(G)$. Moreover, all unipotent elements of $C_R(s)$ are in $C_R(s)^0$.*

Proof. In the group R^0, let us fix a Borel subgroup B containing T. It is clear that all generators mentioned in the lemma belong to $C_R(s)$. Let us prove that $C_R(x)$ is generated by the elements mentioned in the lemma. First, we shall show that the group R (which is not necessarily connected) admits the Bruhat decomposition. Let x be an arbitrary element of R. Then B^x is a Borel subgroup of the group R^0. By Lemma 2.1, there exists an element $s \in R^0$ such that $B^x = B^s$. Then the element xs^{-1} normalizes the subgroup B. The torus $T^{xs^{-1}}$ is a maximal torus of the group B. Since all maximal tori in B are conjugate (lemma 1.4.1), there is an element g in B such that $T^{xs^{-1}} = T^g$. Hence it can be assumed that xs^{-1} normalizes the torus T. Then $xs^{-1} = n_w t$ for some $n_w \in N_R(T)$, $t \in T$. Since t normalizes B and xs^{-1} normalizes B, the element n_w also normalizes B, hence it normalizes the maximal (connected) unipotent subgroup U of the group B.

Since s lies in R^0, it has a Bruhat decomposition, i.e., it can be represented as $u_1 n_{w_1} t_1 v_1$, where $u_1 \in U \cap n_{w_1} U^- n_{w_1}^{-1}$, $n_{w_1} \in N_{R^0}(T)$, $t \in T$, and $v_1 \in U$. Hence the element x can be represented as $x = n_w t u_1 n_{w_1} t_1 v_1$. Since the elements t and n_w normalize U, we obtain the representation of the element x as $x = u_2 n_{w_2} t_2 v_2$, where $u_2 \in U \cap n_{w_2} U^- n_{w_2}^{-1}$, $n_{w_2} \in N_R(T)$, $t_2 \in T$, and $v_2 \in U$. Since this Bruhat decomposition coincides with the Bruhat decomposition of the element x in the group G, this decomposition is unique.

If $x \in C_R(s)$, then, by the Bruhat decomposition, we can write $x = u n_w t v$, where $v \in U$, $t \in T$, and $u \in U \cap n_w U^- n_w^{-1}$. Since s normalizes U, $N(T)$, U^- and commutes with x, the uniqueness of the decomposition implies that each of u, n_w, and v commutes with s. Moreover, since s normalizes each root subgroup U_α, the uniqueness of the decomposition of U into a product of the subgroup U_α ($\alpha > 0$) implies that $\alpha(s) = 1$ whenever u or v contains a nontrivial factor from U_α. Thus x belongs to the group generated by the torus T and by those U_α and n_w which commute with s.

Since T and all U_α with $\alpha(s) = 1$ are connected, the subgroup H generated by them is closed, connected, and normal in $C_R(s)$. Since the Weyl group is finite, we have $|G_R(s) : H| < \infty$, and hence $H = C_R(s)^0$.

Since the roots of the group $C_R(s)$ with respect to the torus T appear in pairs (i.e., if $\alpha(s) = 1$, then $-\alpha(s) = 1$ as well), the group $C_R(s)$ is reductive. Indeed, if $C_R(s)$ has a nontrivial unipotent radical V, then it is normalized by T, and hence it contains a root subgroup U_α. The subgroup V is normalized by the root group $U_{-\alpha}$, which yields an element in V which is not unipotent; we obtain a contradiction.

Since $(|R : R^0|, p) = 1$, all unipotent elements of R lie in R^0, hence all unipotent elements of $C_R(s)$ lie in $C_{R^0}(s)$. The fact that in a connected reductive group R^0 any unipotent element of $C_{R^0}(s)$ lies in $C_{R^0}(s)^0$ is well known (see, for example, [21, 2.2]). ⊠

Let x be a semisimple element in a connected reductive linear algebraic group G. Then, by the previous lemma, $C_G(x)^0$ is a connected reductive subgroup of maximal rank, and $[C_G(x)^0, C_G(x)^0]$ is a semisimple group whose root system is an additively closed subsystem of the root system of G. In what follows, such subgroups will be referred to as *subsystem* subgroups. Since only finite groups are studied in the present paper, the elements of prime order $r \neq p$ are of particular interest. It turns out that the following lemma holds:

Lemma 3.1.3. *[19, 4.1] Let G be a simple connected linear algebraic group over an algebraically closed field of characteristic $p > 0$, and let the element $x \in G$ have prime order $r \neq p$. Suppose that $C' = [C_G(x)^0, C_G(x)^0]$ is a subsystem subgroup. If Δ is the Dynkin diagram of the root system of the group C', then one of the following assertions holds:*

1. *Δ is obtained from the Dynkin diagram of G by removing some vertices;*
2. *Δ is obtained from the extended Dynkin diagram of G by removing a vertex r_i, where $r = c_i$ is the coefficient of r_i in r_0.*

In particular, if r is not a bad number for the group G, then $\dim(Z(C_G(x)^0)) \geqslant 1$.

Concluding this section, let us recall the algorithm of Borel and De Siebenthal for finding all subsystems of a root system Φ; see [6]. Consider the extended Dynkin diagram for the system Φ. The diagrams of all possible subsystems of Φ are obtained by removing several vertices from the extended Dynkin diagram for Φ.

Lemma 3.1.4. *[12, theorem 11.3.2, 14.5.1, and 14.5.2] The following isomorphisms hold:*
1) $A_{n-1}(q) \cong L_n(q) \cong PSL_n(q)$, $n \geqslant 2$;
2) $B_n(q) \cong P\Omega_{2n+1}(q)$, $n \geqslant 3$;
3) $C_n(q) \cong PSp_{2n}(q)$, $n \geqslant 2$;
4) $D_n(q) \cong P\Omega^+_{2n}(q)$, $n \geqslant 4$;
5) ${}^2D_n(q^2) \cong P\Omega^-_{2n}(q)$, $n \geqslant 4$;
6) ${}^2A_n(q^2) \cong PSU_{n+1}(q^2)$, $n \geqslant 2$;
7) $B_2(3) \cong {}^2A_3(2^2)$, $B_n(2^\alpha) \cong C_n(2^\alpha)$, $B_2(q) \cong C_2(q)$.

The following lemma generalizes results obtained in [3], [4], [45], and [46]. Orthogonal groups of low dimension may be found in [45].

Lemma 3.1.5. *Let* $q = p^\alpha$. *Then*
1) $\mathbf{a}_p(GL_n(q)) = q^{[\frac{n^2}{4}]}$;
2) $\mathbf{a}_p(Sp_{2n}(q)) = q^{\frac{n(n+1)}{2}}$;
3) $\mathbf{a}_p(O_{2n+1}(q)) = q^{\frac{n(n-1)}{2}+1}$ *if* $n \geqslant 3$, $p \neq 2$;
4) $\mathbf{a}_p(O^+_{2n}(q)) = q^{\frac{n(n-1)}{2}}$ *if* $n \geqslant 4$;
5) $\mathbf{a}_p(O^-_{2n}(q)) = q^{\frac{(n-2)(n-1)}{2}+2}$ *if* $n \geqslant 5$;
6) $\mathbf{a}_p(O^-_8(q)) = q^6$;
7) $\mathbf{a}_p(U_n(q^2)) = q^{[\frac{n^2}{4}]}$.

Lemma 3.1.6. *[18] Let V be an n-dimension vector space over finite field $GF(q)$, $G \leq GL(V)$, $H \lhd G$, $(|G/H|, q) = 1$, nilpotency class of G/H is at most 2.*
1) $|G/H| < q^n$;
2) If G preserves non degenerate bilinear form f on V, then $|G/H| \leqslant 2^{\varepsilon(n)}\delta^{[\frac{n}{2}]}$, *where*

$$\varepsilon(n) = \begin{cases} 0, \text{if } q \text{ or } n \text{ is even}, \\ 1, \text{if } q \text{ and } n \text{ is odd}, \end{cases} \delta = \begin{cases} 8, \text{if } q\text{=3 or 5}, \\ 1+q, \text{ otherwise}. \end{cases}$$

Lemma 3.1.7. *[18, 1.1] Let V be a finite-dimensional vector space over a field $GF(q)$. Suppose A is a subgroup of $GL(V)$, and $(|A|, q) = 1$. Then V is decomposed into a direct sum of proper irreducible A-submodules.*

Lemma 3.1.8. *[18, 1.2] Let V be a finite-dimensional vector space over $GF(q)$ and f be the non degenerate bilinear form on V. If A is a subgroup of $GL(V)$, A preserves f, and $(|A|, q) = 1$, then $V = C_V(A) \oplus^\perp [V, A]$ is an orthogonal direct sum of A-submodules $C_V(A) = \{v \in V \mid va = v$ for all $a \in A\}$ and $[V, A] = \{va - v \mid v \in V,\ a \in A\}$.*

Lemma 3.1.9. *Suppose that G is a connected reductive linear algebraic group, and A is its closed Abelian subgroup. Then the following assertions hold:*

1. *the group A can be represented as $A_s \times A_u$, i.e., as the direct product of its semisimple part and its unipotent part, respectively;*

2. *There exists a reductive subgroup of maximal rank R of G such that $A \leq R$, $A_u \leq R^0$, and $A_s \cap R^0 = A_{s0} \leq Z(R^0)$.*

3. *If $W_R = N_R(T)/T$, $W_{R0} = N_{R^0}(T)/T$ for some maximal torus T of R, then A_s/A_{s0} embeds into W_R/W_{R0}.*

If A is σ-stable, then R is σ-stable also.

PROOF. 1. Proven in [20, 15.5].

2. Suppose that s is a semisimple element of A_s. Let $R = C_G(s)$. It is clear that $A \leq R$. By lemma 3.1.2, we have $A_u \leq R^0$, and R^0 is a connected reductive subgroup of maximal rank in G. The subgroup R^0 is normal in R; therefore, any element of R normalizes $Z(R^0)$ and hence normalizes $Z(R^0)^0$. By lemma 3.1.2, we have $(|R : R^0|, p) = 1$. If there exists a semisimple element $s_1 \in A_s$ such that $s_1 \in R^0$ but $s_1 \notin Z(R^0)$, then let us consider $C_R(s_1)$. It is clear that $A \leq C_R(s_1)$. By lemma 3.1.2, $C_R(s_1)^0$ is a connected reductive subgroup of maximal rank in G. As above, we have $A_u \leq C_R(s_1)^0$, and any element of $C_R(s_1)$ normalizes $Z(C_R(s_1)^0)^0$. Replacing the group R by $C_R(s_1)$, we obtain a reductive subgroup of maximal rank in G containing A, of smaller dimension. Indeed, the dimension decreases, since the dimension of the identity component decreases. The process described above is finite, since at each step the dimension decreases, and the dimension of G is finite. Note that if A is an Abelian subgroup of a finite group of Lie type, then A consists of fixed points of a Frobenius automorphism σ, and hence the groups obtained at each step of the process described above are σ-invariant. Therefore, if $A \leq G_\sigma$, then the group R is σ-invariant.

3. Further, we have $A_s/A_{s0} \cong A_sR^0/R^0 \leq R/R^0$. By lemma 3.1.2, any element of R can be represented as $n_w x$, where $x \in R^0$; hence the group R/R^0 is isomorphic to $N_R(T)/N_{R^0}(T) \cong W_R/W_{R0}$ for any maximal torus T in R^0. ⊠

For a simple corollary of lemma 3.1.9, note that if $\Phi_R = \Phi$, then $A_s = A_{s0} \leq Z(G)$. Indeed, the reductive group R which is mentioned in the lemma coincides in this case with G (and coincides with R^0), but we have $A_s \cap R^0 = A_s \leq Z(R^0)$.

In lemma 3.1.9 sections of the Weyl group appear, hence it is necessary to find the orders of large Abelian subgroups of the Weyl groups for all simple algebraic groups. Besides, in what follows we shall often encounter the situation in which a semisimple Abelian subgroup is the set of fixed points for a Frobenius automorphism of a torus T of dimension n. To estimate the orders of such subgroups, we shall need the following lemma:

Lemma 3.1.10. *Suppose that S is a σ-invariant torus of dimension n in a connected simple algebraic groups G, where $\sigma = q\sigma_0$ is a Frobenius automorphism. Let S^g be one of its conjugate σ-invariant tori in G. By $X(S)$ denote the group of rational characters of S. Then there exists an element w of the Weyl group W of G such that $X(S)^w \subseteq X(S)$ and the group $(S^g)_\sigma$ is isomorphic to the group $X(S)/(\sigma w - 1)X(S)$. In particular, since the element $\sigma_0 w$ is of finite order, we have $|(S^g)_\sigma| \leqslant (q+1)^n$.*

PROOF. Since the torus S is σ-invariant, its centralizer C in G is a connected σ-invariant reductive subgroup of maximal rank. Then the group C^g is also σ-invariant. The groups C and C^g contain some σ-invariant maximal tori T and T_1 [39, 10.10]; without loss of generality, we may assume that they are also conjugate by g. Let W_1 be the Weyl group of C. In [10, the corollary of Proposition 2] it is asserted that in this case the element $g^\sigma g^{-1}$ lies in $N_G(T)$, and its image under the canonical homomorphism of $N_G(T)$ onto the Weyl group belongs to $N_W(W_1)$. Proposition 8 from [10] asserts that in this case we have $(T^g)_\sigma \cong X(T)/(\sigma w - 1)X(T)$. Since $S \leq Z(C)$, the torus S lies in T. By the bijective correspondence between closed subgroups of the torus T and subgroups of the group $X(T)$ of its characters (see [5, chapter III]), we have $(S^g)_\sigma \cong X(S)/(\sigma w - 1)X(S)$. ⊠

In table 3.1 below we list the orders of the large Abelian subgroups in the Weyl groups of the classical simple groups. The group $W(A_n)$ is isomorphic to S_{n+1}; the orders and the structure of the large Abelian subgroups of this group are found in 2.1, and we cite them in table 3.1 without a proof.

The groups $W(B_n)$ and $W(C_n)$ are isomorphic; hence we shall only consider the group $W(B_n)$. If Φ is a root system of type B_n and $e_1, \dots, e_n$ is an orthonormal basis of the Euclidean space in which the system B_n lies, then Φ can be written as $\{\pm e_i \pm e_j,\ i \neq j, \pm e_i;\ i, j = 1, \dots, n\}$. The Weyl group $W(B_n)$ acts on the set $\{\pm e_1, \dots, \pm e_n\}$ of $2n$ roots. Let A be an Abelian subgroup of $W(B_n)$. Then $I_1, \dots, I_k$ are all A-orbits in $\{\pm e_1, \dots, \pm e_n\}$. Consider the group G of all transformations of the Euclidean space spanned by $e_1, \dots, e_n$ under which the set $\{\pm e_1, \dots, \pm e_n\}$ is invariant. It is clear that $W(B_n) \leq G$. Let us find $\mathbf{a}(G)$ and prove that $\mathbf{a}(G) = \mathbf{a}(W(B_n))$. The proof of theorem 2.1.1 implies that $|A| \leqslant |I_1| \times \cdots \times |I_k|$. Let $f(2n)$ be the order of a large Abelian subgroup in G.

Assume that among the sets $I_1, \dots, I_k$ there are sets of odd order. Without loss of generality, we may assume that one of these sets is the set I_1 and that the basis vector e_1 belongs to I_1. Then the vector $-e_1$ does not lie in I_1. Indeed, assume the converse. The group A acts transitively on I_1; hence there is an element σ in A that takes e_1 to $-e_1$. But in this case the order of the element σ is even, and it can be assumed that σ is a 2-element. Besides, σ does not belong the stabilizer $St_A(I_1)$ of the orbit I_1 in A. Hence its image under the natural homomorphism $\phi\ A \to A/St_A(I_1)$ is also of even order. But the order $|A/St_A(I_1)| = |I_1|$ is odd; we obtain a contradiction. Thus the element $-e_1$ belongs to another set; without loss of generality we may assume that this set is I_2. Since G is a group of linear transformations of Euclidean space, the relation $\sigma(-e_1) = -\sigma(e_1)$ holds for any $\sigma \in G$. Since the group A acts transitively on the sets $I_1, \dots, I_k$, hence follows that if $\sigma \in St_A(I_1)$, then $\sigma \in St_A(I_1) \cap St_A(I_2)$, and for any $v \in I_1$, the vector $-v$ belongs to I_2. Let $m = |I_1| = |I_2|$. Then, by the above arguments, we have $|A| \leqslant m \cdot f(2n - 2m)$.

Consider the group A_1 such that its action on the orbits $I_3, \dots, I_k$ is the same as for the group A, and the set $I_1 \cup I_2$ splits into two-element orbits $\{\pm v\}$ with respect to the action of A_1. By construction, $A_1 \leq G$ and the group A_1 is Abelian. Besides, we have $|A| = m \cdot |St_A(I_1)| < 2^m \cdot |St_A(I_1)| = |A_1|$. Therefore, the set $\{\pm e_1, \dots, \pm e_n\}$ splits into orbits of even order with respect to the action of the group $A \in \mathbf{A}(G)$. By using induction on n, let us show that all orbits are of order 2 or 4 and, therefore, $f(2n) \leqslant 2^n$. Indeed, if the order of an orbit of A is greater than or equal to 6, then $|A| \leqslant 6 \cdot 2^{n-3} < 2^3 \cdot 2^{n-3}$. Hence $f(2n) \leqslant 2^n$. On the other hand, there is an Abelian subgroup of $W(B_n)$ and hence of G that takes e_i to $\pm e_i$ for all i. Its order is equal to 2^n and, therefore, $\mathbf{a}(W(B_n)) = 2^n$.

Note, that $W(D_n)$ is a subgroup of $W(B_n)$ of index 2, so $\mathbf{a}(W(D_n)) \leqslant \mathbf{a}(W(B_n))$.

Table 3.1. Large abelian subgroups of Weil groups

type of G	$\mathbf{a}(W)$
A_n	3^k, if $n = 3k - 1$ $4 \cdot 3^{k-1}$, if $n = 3k$ $2 \cdot 3^k$, if $n = 3k + 1$
$B_n,\ C_n,\ D_n$	$\leqslant 2^n$

We now recall the structure of σ-fixed points of reductive σ-invariant subgroups of

maximal rank in simple linear algebraic groups.

Lemma 3.1.11. *[10, propositions 1, 2, 6, and 8] Suppose that G is a simple connected linear algebraic group, σ is its Frobenius automorphism, $G_1 = M * S$ is a σ-invariant connected reductive subgroup of maximal rank, where M is semisimple and S is the central torus, and G_1^g is a conjugate σ-invariant subgroup. Let Δ_1 be the Dynkin diagram of G_1, and let W_1 be the Weyl group of G_1. Then $g^\sigma g^{-1} \in N_G(G_1) \cap N_G(T) = N$ (for a maximal torus T in G_1) and there is a bijection $\pi\, N \to N_W(W_1)/W_1$.*

Let $\pi(g^\sigma g^{-1}) = w$, and let τ denote the image of the element w under the natural homomorphism $\phi\, N_W(W_1) \to \mathrm{Aut}\ _W(\Delta_1)$ as well as the graph automorphism of M corresponding to the symmetry τ. Then $(M^g)_\sigma \cong M_{\sigma\tau}$.

Let $\overline{P_1}$ be the sublattice of $X(T)$ generated by all rational linear combinations of the roots belonging to Δ_1. Then $(S^g)_\sigma \cong (X(T)/\overline{P_1})/(\sigma w - 1)(X(T)/\overline{P_1})$.

Besides, $|(G_1^g)_\sigma| = |M_\sigma^g| \cdot |S_\sigma^g|$.

Note that in lemma 3.1.11 the group G_σ is considered. In the case when the group G is not simply connected, the finite group $O^{p'}(G_\sigma)$ does not coincide with G_σ, hence the order of $(G_1^g)_\sigma \cap O^{p'}(G_\sigma)$ is less than the number given in the lemma. In this case, we have $G_\sigma = \widehat{H} O^{p'}(G_\sigma)$, where $|\widehat{H} : H| = d_1/d$. Here $\widehat{H}$ is a maximal torus of G_σ, H is a maximal torus of $O^{p'}(G_\sigma)$, d_1 is the order of the center of $O^{p'}(G_\sigma)$, and d is the order of the center of $(G_{sc})_\sigma$. Thus to find the order of the subgroup $(G_1^g)_\sigma$ in the group $O^{p'}(G_\sigma)$, the number given in the lemma should be multiplied by d_1/d. Indeed, a connected reductive subgroup of maximal rank contains a maximal torus of a connected simple linear algebraic group, and hence $(G_1^g)_\sigma = \widehat{H}((G_1^g)_\sigma \cap O^{p'}(G_\sigma))$. Therefore, $|(G_1^g)_\sigma : ((G_1^g)_\sigma \cap O^{p'}(G_\sigma))| = d_1/d$; thus $|(G_1^g)_\sigma \cap O^{p'}(G_\sigma)| = (d_1/d)|(G_1^g)_\sigma|$.

§2 Large abelian subgroups of $A_n(q)$

Lemma 3.2.1. *Let $G \leq GL_n(q)$, with G satisfying the following conditions:*

(1) $G = H_s \times H_u$, where H_s and H_u are, respectively, semisimple and unipotent components of G;

(2) the nilpotency class of H_s does not exceed 2;

(3) H_u is abelian.

Then $|G| \leqslant f_1(n,q)$, where $f_1(n,q) = \max(q^n - 1, (q-1)q^{[\frac{n^2}{4}]})$

Proof. Assume that the statement of the lemma is untrue, and G is its counterexample of minimal order.

Direct calculations show that for every n_1 and n_2 the following inequality holds

$$f_1(n_1,q) f_1(n_2,q) \leqslant f_1(n_1+n_2,q).$$

By lemma 3.1.7, the group H_s is decomposed into irreducible blocks. We can therefore assume that

$$H_s = \begin{pmatrix} H_{s_1} \ldots 0 \\ 0 \ddots 0 \\ 0 \ldots H_{s_k} \end{pmatrix},$$

where all blocks of H_{s_j} are irreducible and numbered in order of increasing dimension. The so structured group H_s indicates that all matrices in H_s are in blockwise-diagonal form,

and each block of H_{s_j} can be conceived of as a semisimple irreducible subgroup of the group $GL_{n_j}(q)$; any H_{s_j} has its nilpotency class at most 2. In addition, $n_1 + \ldots + n_k = n$. Note that now $H_s \leq H_{s_1} \times \ldots \times H_{s_k}$.

Further we may assume that all H_{s_j} coincide and hence $H_s \cong H_{s_1}$. Indeed, $H_u \neq \{1\}$, since otherwise $G = H_s$, and by lemma 3.1.6, we would have $|G| \leqslant q^n - 1 \leqslant f_1(n,q)$, which contradicts the assumption that G is a counterexample. By assumption, $AB = BA$ for any matrices $A \in H_s$ and $B \in H_u$. Write B as follows:

$$B = \begin{pmatrix} B_{1_1} B_{1_2} \ldots B_{1_k} \\ B_{2_1} B_{2_2} \ldots B_{2_k} \\ \ldots \\ B_{k_1} B_{k_2} \ldots B_{k_k} \end{pmatrix},$$

where the dimension of a block coincides with that of a corresponding block in the representation of the group H_s, that is, B_{i_j} has n_j columns and n_i rows. By commutativity, $H_{s_i} B_{i_j} = B_{i_j} H_{s_j}$ and $H_{s_j} B_{j_i} = B_{j_i} H_{s_i}$. By the Schur lemma, the sets $B_{(j,i)} = \{B_{j_i} \mid B \in H_u\}$ and $B_{(i,j)} = \{B_{i_j} \mid B \in H_u\}$ form a division ring. Consequently, if dimensions of the blocks corresponding to groups H_{s_i} and H_{s_j} are different, or these groups are not conjugate, then $B_{(i,j)} = B_{(j,i)} = \{0\}$. In this way, if dimensions of the blocks do not coincide, or some blocks (groups) are not conjugate, then the group H_u, in the same basis as H_s, takes up the following form:

$$H_u = \begin{pmatrix} H_{u_1} 0 \\ \quad 0 H_{u_2} \end{pmatrix}.$$

It follows that $G \leq G_1 \times G_2$, where G_1 and G_2 are subgroups in $GL_{n_1}(q)$ and $GL_{n_2}(q)$ satisfying the assumption of the lemma. Since G is a minimal counterexample, we have

$$|G| \leqslant |G_1| \cdot |G_2| \leqslant f_1(n_1, q) \cdot f_1(n_2, q) \leqslant f_1(n_1 + n_2, q) = f_1(n, q).$$

We have arrived at a contradiction with G being a minimal counterexample.

Thus, not only are dimensions equal for all blocks H_{s_j}, but also all subgroups of H_{s_j} are conjugate. Up to conjugation, therefore, the group H_s has the following form:

$$H_s = \underbrace{\begin{pmatrix} H_{s_1} \ldots 0 \\ \quad 0 \ddots 0 \\ \quad 0 \ldots H_{s_1} \end{pmatrix}}_{k \text{ times}}. \tag{3.1}$$

Consequently, $H_s \cong H_{s_1}$. Assume that the dimension of H_{s_1} equals $\frac{n}{k}$. By Lemma 2.3, $|H_s| = |H_{s_1}| \leqslant (q^{\frac{n}{k}} - 1)$. Since all blocks B_{i_j} of the matrices in H_u form a division ring, and any finite division ring is a field, and, since they lie in $M_{\frac{n}{k}}(q)$, we may assume that these blocks form a subfield in $F_{q^{\frac{n}{k}}}$. The group H_s has the form (3); therefore, we can think that $H_u \leq GL_k(q^{\frac{n}{k}})$. From (1) of Lemma 2.2, it follows that $|H_u| \leqslant q^{\frac{n}{k}[\frac{k^2}{4}]}$. This implies that

$$|G| \leqslant |H_s| \cdot |H_u| \leqslant (q^{\frac{n}{k}} - 1) \cdot q^{\frac{n}{k}[\frac{k^2}{4}]} \leqslant f_1(n, q).$$

Contradiction. $\boxtimes$

By using lemma 3.2.1 one may easy derive the number $\mathbf{a}(A_n(q))$ from table 3.2. Indeed, if A is an abelian subgroup of $A_{n-1}(q)$, then its preimage A_1 under natural homomorphism $\varphi : SL_n(q) \to A_{n-1}(q)$ satisfies to the conditions of lemma 3.2.1. Then the group $A_1Z(GL_n(q))$ satisfies to the conditions of the lemma, hence $|A_1Z(GL_n(q))| < f_1(n,q)$. Thus, $|A| \leqslant f_1(n,q)/((n+1,q-1)(q-1)) \leqslant \mathbf{a}(A_n(q))$.

§3 Large abelian subgroups of $C_n(q)$, $n \geqslant 3$

Lemma 3.3.1. *Suppose $G \leq Sp_{2n}(q)$ and G satisfies the following conditions:*

(1) $G = H_s \times H_u$, where H_s and H_u are, respectively, semisimple and unipotent components of G;

(2) the nilpotency class of H_s is at most 2;

(3) H_u is Abelian.

Then $|G| \leqslant f_2(n,q)$, where $f_2(n,q) = (2,q-1)q^{\frac{n(n+1)}{2}}$, if $n \geqslant 3$, $f_2(2,q) = (2,q-1)(q+1)q^2$, $f_2(1,q) = \max(\delta,(2,q-1)q)$ (δ is taken from lemma 3.1.6).

Proof. Assume that the statement of the lemma is untrue, and G is its minimal counterexample.

Direct calculations show that for every n_1 and n_2 the following inequality holds

$$f_2(n_1,q)f_2(n_2,q) \leqslant f_2(n_1+n_2,q).$$

If the group H_s has no proper submodules then it is a unique irreducible block, $G = H_s$ and G is not a counterexample.

Among all proper H_s-submodules, if any, choose an H_s-submodule U of minimal dimension. Consider $D = C_{H_s}(U)$. If $D = \{1\}$ then H_s acts faithfully on U. By lemma 3.1.6, $|H_s| < q^{\dim\,(U)} = q^k$. As in the proof of the previous lemma, we can obtain $|G| \leqslant |H_s| \cdot |H_u| \leqslant (q^k-1)q^{k[\frac{n^2}{k^2}]} \leqslant f_2(n,q)$. (The latter inequality holds for $n \geqslant 3$ and $k \geqslant 2$.) For $n = 1$, the situation that obtains is the known case where $Sp_2(q) \cong SL_2(q)$. For the case where $n = 2$ and $k = 2$, note that $H_s * Z(GL_4(q))$ satisfies the estimate $|H_s * Z(GL_4(q))| \leqslant (q^2-1)$. Since $|H_s \cap Z(GL_4(q))| \leqslant (2,q-1)$, we have $|H_s| \leqslant (2,q-1)(q+1)$, that is, the inequality $|G| \leqslant f_2(n,q)$ holds in this case, too. If $k = 1$ then $|H_s| = (2,q-1)$. By (2) of lemma 3.1.5, $|H_u| \leqslant q^{\frac{n(n+1)}{2}}$, and so $|G| \leqslant (2,q-1)q^{\frac{n(n+1)}{2}} \leqslant f_2(n,q)$.

We can thus assume $D > \{1\}$. Then $C_V(D)$ and $[V,D]$ are proper nontrivial H_s-submodules. By lemma 3.1.8, $V = C_V(D) \oplus^{\perp} [V,D]$; hence, H_s can be represented as a subgroup in the group $H_{s_1} \times H_{s_2}$, where H_{s_1} and H_{s_2} are semisimple class 2 nilpotent subgroups in the groups $Sp_{2n_1}(q)$ and $Sp_{2n_2}(q)$, respectively. Repeating the above argument for H_{s_1} and H_{s_2} yields $H_s \leq H_{s_1} \times \ldots \times H_{s_k}$, where all H_{s_j} are irreducible, semisimple, nilpotent subgroups of class 2 in $Sp_{2n_j}(q)$.

We make use of the estimate obtained in lemma 3.2.1 and the fact that H_u can be treated as a subgroup of $GL_k(q^{\frac{2n}{k}})$, that is, $|H_u| \leqslant q^{\frac{2n}{k}[\frac{k^2}{4}]}$. As indicated above, H_s has the form

$$H_s = \underbrace{\begin{pmatrix} H_{s_1} & & 0 \\ & \ddots & \\ 0 & & H_{s_1} \end{pmatrix}}_{k\text{ times}}$$

that is, $H_s \cong H_{s_1}$, and hence $|H_s| \leqslant \delta^{\frac{n}{k}}$ by lemma 3.1.6. It follows that $|G| \leqslant \delta^{\frac{n}{k}} \cdot q^{\frac{2n}{k}[\frac{k^2}{4}]} \leqslant q^{\frac{n^2}{2}+\frac{n}{2}} \leqslant f_2(n,q)$ for $n \geqslant 3$. For $n=2$ and $k=2$, $|G| \leqslant \delta \cdot q^2 \leqslant f_2(n,q)$. Contradiction. $\boxtimes$

The number $\mathbf{a}(C_n(q))$ from table 3.2 may be obtained like the number $\mathbf{a}(A_n(q))$.

§4 Large abelian subgroups of ${}^2A_n(q^2)$

Till the end of the chapter we use notations of lemma 3.1.11. Let $\overline{G}$ be a simple connected linear algebraic group with root system Φ, σ a Frobenius automorphism ($\sigma = q\sigma_0$, $q = p^\alpha$), and $G = O^{p'}(\overline{G}_\sigma)$. Let $A = A_s \times A_u$ be an abelian subgroup of G. We may assume, that $A_s \nleqslant Z(G)$. By lemma 3.1.9 there exists connected reductive σ-stable subgroup R (R^0 in notations of lemma 3.1.9) of maximal rank of $\overline{G}$, containing $A_0 = A_u \times A_{s0}$ ($A_{s0} = A_s \cap R$). More over, A/A_0 is isomorphic to a section of the $W(\Phi)$, $A_{s0} \leq Z(R)$. Suppose, that $S = Z(R)^0$.

Lemma 3.4.1. *[9, proposition 8] Let $\overline{G}$ be a group of type A_n and $\overline{G}_\sigma$ be the twisted form of $\overline{G}$. Let G_1 be a σ-stable reductive subgroup of maximal rank in $\overline{G}$, corresponding to a partition λ of $n+1$. Let G_1^g be a σ-stable reductive subgroup of $\overline{G}$ obtained by twisting G_1 by an element $w \in W$ defined by $\pi(g^\sigma g^{-1}) = w$. Suppose w maps to τ under the homomorphism $N_W(W_1) \to Aut_W(\Delta_1)$. Let n_i be the number of parts of λ equal to i, so that $Aut_W(\Delta_1) \cong S_{n_2} \times S_{n_3} \times \dots$. Suppose $\sigma_0\tau$ ($|\sigma_0| = 2$) gives rise to partitions $\mu^{(2)}, \mu^{(3)}, \dots$ of $n_2, n_3, \dots$ respectively. Then the simple components of the semisimple group $(M^g)_\sigma$ are of type $A_{i-1}(q^{\mu_j^{(i)}})$ for $\mu_j^{(i)}$ even and of type ${}^2A_{i-1}(q^{2\mu_j^{(i)}})$ for $\mu_j^{(i)}$ odd.*

The order of the toral part $(S^g)_\sigma$ is given by

$$(q+1)|(S^g)_\sigma| = \prod_{i,j\ \mu_j^{(i)} even} (q^{\mu_j^{(i)}}-1) \prod_{i,j\ \mu_j^{(i)} odd} (q^{\mu_j^{(i)}}+1).$$

Proof. Since the group A/A_0 is isomorphic to a section of the Weil group $W(A_n) \cong S_{n+1}$, by table3.1 we have that $|A/A_0| \leqslant 3^{(n+1)/3}$. Therefore, by lemmas 3.1.5, 3.4.1 and table 3.1, we may obtain the following bound:

$$(q+1)|A| \leqslant \frac{d_1}{d} 3^{(n+1)/3} \prod_{i,j\ \mu_j^{(i)}\ even} (q^{\mu_j^{(i)}}-1) \prod_{i,j\ \mu_j^{(i)}\ odd} (q^{\mu_j^{(i)}}+1) \prod_{i,j} (i, q^{\mu_j^{(i)}}-1) q^{\mu_j^{(i)}[i^2/4]}. \quad (3.2)$$

Set $S = Z(R)^0$. Since there doe not exist bad primes for a root system A_n (in view of lemma 3.1.3), it follows that $\dim(S) \geqslant 1$. Consider the case, when $\dim(S) = 1$. Then the group R has only two simple components, otherwise the dimension of the central torus is greater than 1. Hence, $R = A_{m_1-1}(K) * A_{m_2-1}(K) * S$ and $m_1 + m_2 = n+1$. We prove that $R = C_{A_n(K)}(S)$ in this case. The embedding $R \leq C_{A_n(K)}(S)$ is evident. The centralizer of torus in connected reductive algebraic group is again connected. The unique subsystem connected subgroup such that it contains $A_{n_1-1}(K) * A_{n_2-1}(K)$ is the group $A_n(K)$. Since its center is of dimension 0, it cannot coincide with S. Therefore, $A_n(K) \neq C_{A_n(K)}(S)$ and $R = C_{A_n(K)}(S)$. Thus, any element of $A_n(K)$, centralizing S, lies in R.

Assume that $A_s \neq A_{s0}$. Since A_s normalizes S and $A_{s0} \in R$, we obtain the action of A_s/A_{s0} on the group of characters $X(S)$ of S by the following low. If $x \in A_s/A_{s0}$ and

$\chi \in X(S)$, then for any $s \in S$ we have $\chi^x(s) = \chi(s^x)$. Here s^x defines the conjugation of s by any representative of x in A_s. Clearly, that this conjugation does not depend on representative, since A_{s0} centralizes S. The action of x is an automorphism of $X(S)$. Indeed, x preserves multiplication, so x is a homomorphism of $X(S)$. Since there exists inverse of x, x is a bijection and, hence, an automorphism of $X(S)$. The dimension of S equals 1, so $X(S)$ is isomorphic to $\mathbb{Z}$. The unique nontrivial automorphism of $\mathbb{Z}$ is an automorphism of order 2. Thus, if A_s/A_{s0} is nontrivial, then its order equals 2.

More over, in this case $|A_{s0} \cap S| \leqslant 2$. Actually, A_s is abelian, so any element of A_{s0} is x-stable. Let χ be an element of $X(S)$ such that $\langle\chi\rangle = X(S)$. By definition, it follows that for every $t \in A_{s0} \cap S$ the identity holds $\chi(t) = \chi^{-1}(t^x) = \chi^{-1}(t) = \chi(t^{-1})$, i. e. $t^{-1} = t$. In particular, $|A_{s0} \cap S| \leqslant 2$, since every finite subgroup of X is cyclic. Since $\frac{d_1}{d} \leqslant 1$, we may substitute the factor $\frac{d_1}{d}$ by 1. The factor $3^{(n+1)/3}$ appears, when we bound the order of A/A_0. Since $|A/A_0| = 2$, the factor $3^{(n+1)/3}$ is substituted by 2. By above we may take $|S \cap A_0| \leqslant 2$ instead of $|S|$. Thus, in view of (3.2) we have that either

$$|A| \leqslant 4 \cdot (m_1, q+1)q^{[m_1^2/4]} \cdot (m_2, q+1)q^{[m_2^2/4]},$$

or

$$|A| \leqslant 4 \cdot ((n+1)/2 - 1, q^2 - 1)q^{2[(n+1)/16]}$$

(second case arises, when $m_1 = m_2$, i. e. $n+1$ is even). We show that if $n \neq 2$ and $q \neq 3$, than in both cases $|A| \leqslant q^{[(n+1)^2/4]}$.

If $n \geqslant 14$, than we have that

$$\begin{array}{llll} & 4 \cdot (m_1, q+1)q^{[m_1^2/4]} \cdot (m_2, q+1)q^{[m_2^2/4]} & \leqslant 4 \cdot (q+1)^2 \cdot q^{m_1^2/4+m_2^2/4} & \leqslant \\ \leqslant & q^6 \cdot q^{m_1^2/4+m_2^2/4} & \leqslant \quad q^{m_1^2/4+m_2^2/4+7-1} & \leqslant \\ \leqslant & q^{m_1^2/4+m_2^2/4+n/2-1} & \leqslant \quad q^{m_1^2/4+m_2^2/4+m_1m_2/2-1} & \leqslant \\ \leqslant & q^{[(n+1)^2/4]}. & & \end{array}$$

If $8 \leqslant n \leqslant 13$, $m_1, m_2 > 1$, than

$$\begin{array}{llll} & 4 \cdot (m_1, q+1)q^{[m_1^2/4]} \cdot (m_2, q+1)q^{[m_2^2/4]} & \leqslant 4 \cdot (q+1)^2 \cdot q^{m_1^2/4+m_2^2/4} & \leqslant \\ \leqslant & q^6 \cdot q^{m_1^2/4+m_2^2/4} & \leqslant \quad q^{m_1^2/4+m_2^2/4+8-2} & \leqslant \\ \leqslant & q^{m_1^2/4+m_2^2/4+n-2} & \leqslant \quad q^{m_1^2/4+m_2^2/4+m_1m_2/2-1} & \leqslant \\ \leqslant & q^{[(n+1)^2/4]}. & & \end{array}$$

If $6 \leqslant n \leqslant 7$, $q > 2$, and $m_1, m_2 > 1$, than

$$\begin{array}{llll} & 4 \cdot (m_1, q+1)q^{[m_1^2/4]} \cdot (m_2, q+1)q^{[m_2^2/4]} & \leqslant 4 \cdot (q+1)^2 \cdot q^{m_1^2/4+m_2^2/4} & \leqslant \\ \leqslant & q^4 \cdot q^{m_1^2/4+m_2^2/4} & \leqslant \quad q^{m_1^2/4+m_2^2/4+6-2} & \leqslant \\ \leqslant & q^{m_1^2/4+m_2^2/4+n-2} & \leqslant \quad q^{m_1^2/4+m_2^2/4+m_1m_2/2-1} & \leqslant \\ \leqslant & q^{[(n+1)^2/4]}. & & \end{array}$$

If $8 \leqslant n \leqslant 13$, $q > 2$, and $m_1 = 1$, than

$$\begin{array}{llll} & 4 \cdot (n, q+1)q^{[n^2/4]} & \leqslant 4 \cdot (q+1) \cdot q^{n^2/4} & \leqslant \\ \leqslant & q^3 \cdot q^{n^2/4} & \leqslant \quad q^{n^2/4+4-1} & \leqslant \\ \leqslant & q^{[(n+1)^2/4]}. & & \end{array}$$

If $n = 7$, $q > 2$, and $m_1 = 1$, than $4 \cdot (7, q+1)q^{12} < q^{16}$. If $n = 6$, $q > 2$, and $m_1 = 1$, than $4 \cdot (6, q+1)q^9 < q^{12}$. If $n = 5$, $q > 2$, if $m_1 = 1$, than $4 \cdot (5, q+1)q^6 < q^9$. If $n = 5$, $q > 2$, $m_1 = 2$, $m_2 = 4$, than $4 \cdot (2, q+1)q(4, q+1)q^4 < q^9$. If $n = 5$, $q > 2$, $m_1 = 3$, $m_2 = 3$, than $4 \cdot (3, q+1)q^2(3, q+1)q^2 < q^9$. In other cases we proceed in similar way. Below in analogous situations we omit calculations.

Consider ${}^2A_2(3^2)$, if $m_1 = 1, m_2 = 2$. The order of some of its abelian subgroup equals 16. This is some maximal torus, isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_4$. The torus is unique up to conjugation. The group ${}^2A_2(3^2)$ is contained in $A_2(K)$, where K is an algebraically closed field of characteristic 3. In this case $A_2(K)$ is simply connected, i. e. $\Gamma_\pi = \Gamma_{sc}$. R equals $M * S$, where M is a semisimple group of type A_1, S a torus of dimension 1, and $M \cap S$ is finite. If M is group of adjoint type, than its center is trivial, so $|A| \leqslant |S| \cdot |\mathbf{a}_u(A_1(3))| \leqslant 4 \cdot 3 < 16$. If M is simply connected. than its center is nontrivial, its order equals 2. We show that the intersection $M \cap S$ coincides with the center of M. Indeed, $M = \langle X_r, X_{-r} \rangle$, where r is some root of root system A_2. Let P be the lattice, generated by r, $\overline{P} = \Gamma_{sc} \cap \mathbb{Q}\langle r \rangle$. Then

$$S = \overline{P}^{\perp} = \{t | \forall l \in \overline{P}, t^l = 1\}.$$

In algebraic group of type A_2 the identity holds $|\Gamma_{sc} : \Gamma_{ad}| = 3$. The group $\overline{P}$ is cyclic, assume that $\overline{P} = \langle \chi \rangle$. Then either $\chi \in P$, or $\chi^3 \in P$, since $P = \overline{P} \cap \Gamma_{ad}$. Let x be a nontrivial element in $Z(M)$. Since $|Z(M)| = 2$, we have $x^2 = 1$. Now $x^r = 1$, hence $1 = x^{\chi^3} = x^{\chi^2} x^{\chi} = (x^2)^{\chi} x^{\chi} = x^{\chi} = 1$, therefore $x \in S$. So, $|A| \leqslant 4 \cdot 3 < 16$.

Assume, that $A_s = A_{s0}$. By lemma 3.1.9 we have that A_{s0} is contained in the center of R. It follows that we may estimate the order of A as follows $|A| \leqslant (q+1)(m_1, q+1)q^{[m_1^2/4]}(m_2, q+1)q^{[m_2^2/4]}$, or $|A| \leqslant (q-1)((n+1)/2, q^2-1)q^{2[(n+1)^2/16]}$. The inequality holds $|A| \leqslant q^{[(n+1)^2/4]}$, for $n \geqslant 5$. Cases $n = 2$, $n = 3$ we consider separately. If $n = 4$ we need to consider only following cases $(m_1 = 1,\ m_2 = 4,\ q = 3)$ and $(m_1 = 2,\ m_2 = 3,\ q = 2)$.

Consider the case $n = 2$. If $(3, q+1) = 3$, then either the center of ${}^2A_2(q^2)$ is nontrivial, so $|A| \leqslant (q+1)(m_1, q+1)q^{[m_1^2/4]}(n_2, q+1)q^{[m_2^2/4]} \leqslant 3q^2$; or $|A| \leqslant \frac{1}{3}(q+1)(m_1, q+1)q^{[m_1^2/4]}(m_2, q+1)q^{[m_2^2/4]} \leqslant q^2$. If $(3, q+1) = 1$, then ${}^2A_2(q^2)$ is universal and its center is trivial. Large abelian subgroup in this case coincide with a maximal torus of order $(q+1)^2$, isomorphic to $\mathbb{Z}_{q+1} \times \mathbb{Z}_{q+1}$, and unique up to conjugation. We again have to consider the case $m_1 = 1, m_2 = 2$. We may prove, that $M \cap S$ equals $Z(M)M$. The proof follows the proof for group ${}^2A_2(3^2)$. Hence, $|A| \leqslant (q+1)q < (q+1)^2 \leqslant \mathbf{a}({}^2A_2(q^2))$, where the number $\mathbf{a}({}^2A_2(q^2))$ is taken from table 3.2.

Suppose, that $n = 3$. If $m_1 = m_2 = 2$, then $|A| \leqslant (q+1)(2, q+1)(2, q+1)q^2 \leqslant q^4$ for all q such that $q \neq 3$. In ${}^2A_3(3^2)$ the center is equal to 1,2, or 4. If the center is trivial, then ${}^2A_3(3^2)$ is of adjoin type. So $|A| \leqslant \frac{1}{4} 4 \cdot 2 \cdot 2 \cdot 3^2 \leqslant 3^4$. If the center has order 2, then $|A| \leqslant \frac{1}{2} 4 \cdot 2 \cdot 2 \cdot 3^2 \leqslant 2 \cdot 3^4$. If, at least, the center has order 4, then $|A| \leqslant 4 \cdot 2 \cdot 2 \cdot 3^2 \leqslant 4 \cdot 3^4$. If $m_1 = 1, m_2 = 3$, then we have, that $M \cap S$ equals $Z(M)$ (we may prove this fact like we prove that $M \cap S = Z(M)$ in case ${}^2A_2(3^2)$). Therefore, $|A| \leqslant (q+1)q^2 \leqslant q^4$.

Two remaining cases ${}^2A_4(2^2)$ and ${}^2A_4(3^2)$ are considered in similar way. Thus we study the case, when the dimension of the central torus equals 1.

Suppose that the $\dim S \geqslant 2$. For all m and k the following inequality holds:

$$(q^k + (-1)^{k+1})(m, q^k + (-1)^{k+1})q^{k[m^2/4]} \leqslant (q+1)(mk, q+1)q^{[m^2k^2/4]} \tag{3.3}$$

More over, if $1 \leqslant m_1 \leqslant m_2$, except $m_1 = 1$, $m_2 = 1$; $m_1 = 1$, $m_2 = 2$, and $m_1 = 1$,

$m_2 = 3$, $q = 2$, the following inequality holds

$$(q+1)(m_1, q+1)q^{[m_1^2/4]}(q+1)(m_2, q+1)q^{[m_2^2/4]} \leqslant (q+1)(m_1+m_2, q+1)q^{[(m_1+m_2)^2/4]} \quad (3.4)$$

By using (3.3) and (3.4), expression (3.2) is reduced to one of the following forms

$$(q+1)^2 3^{(n+1)/3}(m_1, q+1)q^{[m_1^2/4]}(m_2, q+1)q^{[m_2^2/4]}(m_3, q+1)q^{[m_3^2/4]} \quad (m_1+m_2+m_3 = n+1) \quad (3.5)$$

$$(q+1)^{n-1}3^{(n+1)/3}(2, q+1)q \quad (3.6)$$

$$(q+1)^{n-3}3^{(n+1)/3}(2, q+1)^2 q^2 \quad (3.7)$$

$$3^{n+(n+1)/3}2^2 \quad (3.8)$$

$$3^{n+(n+1)/3-2}2^4 \quad (3.9)$$

Last two cases we consider only for $q = 2$. If $n \geqslant 5$, then expressions (3.5)–(3.9) do not exceed the number $\mathbf{a}(^2A_5(q^2))$ from the table 3.2. Thus we only need to consider groups $^2A_n(q^2)$ for $n \leqslant 4$ (if $n = 4$ we need to consider only $^2A_4(2^2)$). In all cases we proceed in analogous way, so we consider the case $n = 3$. If $n = 3$, then we obtain two possibilities for a connected reductive group R. The group R is either maximal torus, or equals commutative product of central torus of dimension 2 and of $A_1(K)$.

Assume that R is a maximal torus, i. e. $A = A_s$. Then R is a homomorphic image of diagonal group under natural homomorphism $SL_4(K) \to SL_4(K)/Z$, where $Z \leq Z(SL_4(K))$. Since A_s normalizes R, we may define an action of A_s/A_{s0} on R by $s^{\overline{x}} = s^x$, where $s \in R$, $x \in A_s$, and $\overline{x} \in A_s/A_{s0}$ is the image of x in A_s/A_{s0} under natural homomorphism. Elements of A_s/A_{s0} permute diagonal elements of R. Under the action of A_s/A_{s0} the set $\{1, 2, 3, 4\}$ (the set of all diagonal places) is partitioned on orbits. Since A_s is abelian, then every element of A_s/A_{s0} centralizes A_{s0}. Hence, if s_1 and s_2 are in the same orbit, then $s_1 = zs_2$, where

$$\begin{pmatrix} z & 0 & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & z \end{pmatrix} \in Z.$$

Therefore there exists subgroup A_{s1} of A_{s0} of index at most $|Z|$ (and $|Z| \leqslant 4$) such that A_{s_1} is contained in a torus of dimension equals number of orbits of A_s/A_{s0} minus 1. If there exist four orbits, then $A_s = A_{s0}$ and by lemma 3.1.10 we have $|A| \leqslant (q+1)^3 \leqslant \mathbf{a}(^2A_3(q^2))$. If there exist three orbits, then $|A_s/A_{s0}| = 2$ and $|A_{s0}/A_{s_1}| \leqslant (4, q+1)$, $|A_{s1}| \leqslant (q+1)^2$. So $|A_s| \leqslant 2(4, q+1)(q+1)^2 \leqslant \mathbf{a}(^2A_3(q^2))$. If there exist two orbits, then $|A_s/A_{s0}| \leqslant 4$, $|A_{s0}/A_{s1}| \leqslant (4, q+1)$, $|A_{s1}| \leqslant q+1$, hence $|A_s| \leqslant 4(4, q+1)(q+1) \leqslant \mathbf{a}(^2A_3(q^2))$. If there exists just one orbit, then $|A_s/A_{s0}| \leqslant 4$, $A_{s0}/A_{s1} = A_{s0} \leqslant Z$, so $|A_s| \leqslant 4(4, q+1) \leqslant \mathbf{a}(^2A_3(q^2))$.

Suppose that R is a commutative product of torus of dimension two and simple connected group M of type A_1. By lemma 3.1.9, $A_s \leq N(R)$. By [10, proposition 4] it follows that $N(R)/R$ is isomorphic to $K \leftthreetimes (W_R^{\perp})$, where K is the group of symmetries of Dynkin diagram of M, group $W_R^{\perp}$ is defined in [10]. Since rank M equals 1 and dim $Z(R^0)^0 = 2$, we obtain that K is trivial and $W_R^{\perp} \leq S_3$. Therefore, $|A_s/A_{s0}| \leqslant 3$, in particular A_s/A_{s0} is cyclic. We may find A_s-stable maximal torus T of R (like in

the proof of lemma 3.1.2). S is contained on all maximal torus, hence it is contained in T. So A_s/A_{s0} permutes diagonal elements of S (since T is diagonal group). If A_s/A_{s0} is nontrivial, then A_{s0} contains a connected subgroup B of index at most $|Z|$ such that $\dim B \leqslant 1$. So $|A| = |A_s| \cdot |A_u| \leqslant 4(4, q+1)(q+1)q \leqslant \mathbf{a}({}^2A_3(q^2))$. If $A_s = A_{s0}$, then $|A| \leqslant (q+1)^2 \cdot q \leqslant \mathbf{a}({}^2A_3(q^2))$. $\boxtimes$

§5 Large abelian subgroups of $D_n(q)$ and ${}^2D_n(q^2)$

If algebraic group G is of type D_l, then lemma 3.1.11 gives us the following result.

Lemma 3.5.1. *[9, proposition 10] Let G be a group of type D_l over an algebraically closed field of characteristic p and let G_1 be a σ-stable reductive subgroup of maximal rank in G determined by a pair of partitions λ, μ with $|\lambda| + |\mu| = l$. Let k_i be the number of pairs of λ equal to i, and let n_i be the number of pairs of μ equal to i ($n_1 = 0$). Then*

$$Aut_W(\Delta_1) \cong \begin{cases} S_{k_2} \times (\mathbb{Z}_2 \wr S_{k_3}) \times (\mathbb{Z}_2 \wr S_{k_4}) \times \ldots \times (\mathbb{Z}_2 \wr S_{n_2}) \times (\mathbb{Z}_2 \wr S_{n_3}) \times \ldots \ if\ k_1 > 0, \\ a\ subgroup\ of\ index\ 2\ in\ this\ if\ k_1 = 0. \end{cases}$$

Let G_1^g be a σ-stable reductive subgroup of G, obtained by twisting G_1 by an element $w \in W$ defined by $\pi(g^\sigma g^{-1}) = w$. Suppose w maps to $\tau \in Aut_W(\Delta_1)$. Suppose $\sigma_0\tau$ ($\sigma = q\sigma_0$) gives rise to partitions $\xi^{(i)}$, $\eta^{(i)}$ with $|\xi^{(i)}| + |\eta^{(i)}| = k_i$ (where $\eta^{(i)}$ is vacuous $i = 2$), and pairs of partitions $\zeta^{(i)}$, $\omega^{(i)}$ with $|\zeta^{(i)}| + |\omega^{(i)}| = n_i$. Then the simple components of the group $(M^g)_\sigma$ are of type $A_{i-1}(q^{\xi_j^{(i)}})$, ${}^2A_{i-1}(q^{2\eta_j^{(i)}})$, $D_i(q^{\zeta_j^{(i)}})$, ${}^2D_i(q^{2\omega_j^{(i)}})$ with one component for each part of each $\xi^{(i)}$, $\eta^{(i)}$, $\zeta^{(i)}$, $\omega^{(i)}$.

If $k_1 = 0$ the total number of components of type ${}^2A_{i-1}$ with i odd, and type ${}^2D_i$ is even if $\sigma_0 = 1$, and odd, if $\sigma_0 \neq 1$.

The order of the torus $(S^g)_\sigma$ is given by $|(S^g)_\sigma| = \prod_{i,j}(q^{\xi_j^{(i)}} - 1)\prod_{i,j}(q^{\eta_j^{(i)}} + 1)$.

In view of table 3.1 and lemmas 3.1.5, 3.5.1 the following inequality holds.

$$\begin{aligned} |A| \leqslant 2^l \prod_{i,j}(q^{\xi_j^{(i)}} - 1)\prod_{i,j}(q^{\eta_j^{(i)}} + 1)\prod_{i,j}(i, q^{\xi_j^{(i)}} - 1)q^{\xi_j^{(i)}[i^2/4]}\prod_{i,j}(i, q^{\eta_j^{(i)}} + 1)q^{\eta_j^{(i)}[i^2/4]}\times \\ \times \prod_{i,j}(4, q^{\zeta_j^{(i)}} - 1)q^{\zeta_j^{(i)}\frac{i(i-1)}{2}}\prod_{4,j}(4, q^{\omega_j^{(4)}} + 1)q^{\omega_j^{6(4)}}\prod_{i>5,j}(4, q^{\omega_j^{(i)}} + 1)q^{\omega_j^{(i)}\frac{(i-1)(i-2)+2}{2}}. \end{aligned} \tag{3.10}$$

For any $m \geqslant 3$ the inequality holds:

$$(q^k - 1)(\frac{m}{k}, q^k - 1)q^{k[m^2/4k^2]} \leqslant q^{\frac{m(m-1)}{2}}, \tag{3.11}$$

and for any $m \geqslant 4$ the inequality holds:

$$(q^k + 1)(\frac{m}{k}, q^k + 1)q^{k[m^2/4k^2]} \leqslant q^{\frac{m(m-1)}{2}}. \tag{3.12}$$

For any m the inequalities hold:

$$(4, q^k - 1)q^{k\frac{m/k(m/k-1)}{2}} \leqslant q^{\frac{m(m-1)}{2}}, \tag{3.13}$$

$$(4, q^k + 1)q^{k\frac{(m/k-1)(m/k-2)+2}{2}} \leqslant q^{\frac{(m-1)(m-2)+2}{2}}. \tag{3.14}$$

More over, for all m_1, m_2 we have:

$$(4, q-1)q^{\frac{m_1(m_1-1)}{2}} q^{\frac{m_2(m_2-1)}{2}} \leqslant q^{\frac{(m_1+m_2)(m_1+m_2-1)}{2}}, \tag{3.15}$$

$$(4, q+1)q^{\frac{(m_1-1)(m_1-2)+2}{2}} q^{\frac{(m_2-1)(m_2-2)+2}{2}} \leqslant q^{\frac{(m_1+n_2-1)(m_1+m_2-2)+2}{2}}, \tag{3.16}$$

and

$$(4, q+1)q^6 \cdot q^{\frac{(m-1)(m-2)+2}{2}} \leqslant q^{\frac{(m+4-1)(m+4-2)+2}{2}}. \tag{3.17}$$

Assume $\sigma_0 = 1$, i. e. $O^{p'}((D_l(K))_\sigma)$ is a split group. Then, by using inequalities (3.11)–(3.16), we reduce right hand of (3.10) to the following:

$$\begin{aligned} 2^l(2(q-1)q)^{m_1}(2(q+1)q)^{m_2}((3,q+1)(q+1)q^2)^{m_3}(4,q-1)q^{\frac{m_4(m_4-1)}{2}} \times \\ \times (4,q-1)q^{\frac{m_5(m_5-1)}{2}}(4,q+1)q^{\frac{(m_6-1)(m_6-2)+2}{2}}, \end{aligned} \tag{3.18}$$

where $m_1+m_2+m_3+m_4+m_5+m_6 = l$, $m_4, m_5, m_6 \geqslant 2$, (some of m_i may be equal to 0). For all m_1, m_2, m_3, m_4, m_5, and m_6 right hand of (3.18) is less than or equal to $\mathbf{a}(D_n(q))$. $\mathbf{a}(D_n(q))$ is taken from table 3.2.

Let $\sigma_0 \neq 1$, i. e $O^{p'}((D_l(K))_\sigma)$ is a twisted group. By using inequalities (3.11)–(3.17) we reduce right hand of (3.10) to (3.18). By lemma 3.5.1, we have that $m_6 \neq 0$. Again, for all m_1, m_2, m_3, m_4, m_5, and m_6 right hand of (3.18) is less than or equal to $\mathbf{a}(^2D_2(q^2))$, where $\mathbf{a}(^2D_2(q^2))$ is taken from table 3.2.

§6 Large abelian subgroups in $B_n(q)$, where q is odd

If algebraic group G is of type B_l, then lemma 3.1.11 gives us the following result.

Lemma 3.6.1. *[9, proposition 11] Let G be a group of type B_l over an algebraically closed field of characteristic $p \neq 2$. Let G_1 be a reductive subgroup of maximal rank in G, defined by a triple (λ, μ, ν), where λ, μ are partitions, ν is a nonnegative integer, and $|\lambda| + |\mu| + \nu = l$. Let k_i be the number of parts of λ equal to i, and n_i be the number of parts of μ, equal to i ($n_1 = 0$). Then*

$$Aut_W(\Delta_1) \cong S_{k_2} \times (\mathbb{Z}_2 \wr S_{k_3}) \times (\mathbb{Z}_2 \wr S_{k_4}) \times \ldots \times (\mathbb{Z}_2 \wr S_{n_2}) \times (\mathbb{Z}_2 \wr S_{n_3}) \times \ldots.$$

Let G_1^g be a σ-stable reductive subgroup of G obtained by twisting G_1 by an element $w \in W$ defined by $\pi(g^\sigma g^{-1}) = w$. Suppose w maps to $\tau \in Aut_W(\Delta_1)$. Suppose τ gives rise to a pair of partitions $\xi^{(i)}$, $\eta^{(i)}$, with $|\xi^{(i)}| + |\eta^{(i)}| = k_i$ and a pair of partitions $\zeta^{(i)}$, $\omega^{(i)}$ with $|\zeta^{(i)}| + |\omega^{(i)}| = n_i$ such that the parts of these partitions give the lengths of the positive and negative cycles in the components of τ. Then the simple components of the group $(M^g)_\sigma$ are of type $A_{i-1}(q^{\xi_j^{(i)}})$, $^2A_{i-1}(q^{2\eta_j^{(i)}})$, $D_i(q^{\zeta_j^{(i)}})$, $^2D_i(q^{2\omega_j^{(i)}})$, $B_\nu(q)$ with one component for each part of each $\xi^{(i)}$, $\eta^{(i)}$, $\zeta^{(i)}$, $\omega^{(i)}$.

The order of the torus $(S^g)_\sigma$ is given by $|(S^g)_\sigma| = \prod_{i,j}(q^{\xi_j^{(i)}} - 1)\prod_{i,j}(q^{\eta_j^{(i)}} + 1)$.

From table 3.1 and lemmas 3.1.5, 3.6.1 we obtain the following inequality:

$$|A| \leqslant 2^l \prod_{i,j}(q^{\xi_j^{(i)}}-1)\prod_{i,j}(q^{\eta_j^{(i)}}+1)\prod_{i,j}(i,q^{\xi_j^{(i)}}-1)q^{\xi_j^{(i)}[i^2/4]}\prod_{i,j}(i,q^{\eta_j^{(i)}}+1)q^{\eta_j^{(i)}[i^2/4]}\times$$
$$\times\prod_{i,j}(4,q^{\zeta_j^{(i)}}-1)q^{\zeta_j^{(i)}\frac{i(i-1)}{2}}\prod_{4,j}(4,q^{\omega_j^{(4)}}+1)q^{6\omega_j^{(4)}}\prod_{i>5,j}(4,q^{\omega_j^{(i)}}+1)q^{\omega_j^{(i)}\frac{(i-1)(i-2)+2}{2}}2q^{\frac{\nu(\nu-1)+1}{2}}. \tag{3.19}$$

By using (3.11)–(3.17), right hand of (3.19) is reduced to:

$$2^l(2(q-1)q)^{m_1}(2(q+1)q)^{m_2}((3,q+1)(q+1)q^2)^{m_3}(4,q-1)q^{\frac{m_4(m_4-1)}{2}}\times$$
$$\times(4,q+1)q^{\frac{(m_5-1)(m_5-2)+2}{2}}2q^{\frac{m_6(m_6-1)+1}{2}}, \tag{3.20}$$

where $m_1+m_2+m_3+m_4+m_5+m_6=l$, $m_4,m_5\geqslant 2$. For all m_1,m_2,m_3,m_4,m_5, and m_6 right hand of (3.20) is not greater than $\mathbf{a}(B_l(q))$, where $\mathbf{a}(B_l(q))$ is taken from table 3.2.

§7 Large abelian subgroups in $B_2(2^n)$

Since the characteristic of K equals 2, $B_2(K)$ does not contains semisimple elements of even order. So order of semisimple elements in this case is a good number. We also have that $Z(B_2(K))$ is trivial. More over, $W(B_2)$ is a 2-group, hence $A_0=A$. Therefore, central torus S of R has dimension at least 1 and $A\leq R$. Thus we have only two possibilities.

Let $\dim S=1$. Then $R=S*A_1(K)$ and $Z(A_1(K))$ is trivial. Therefore, by lemmas 3.1.5 and 3.1.10, we get $|A|\leqslant(q+1)q\leqslant q^3=\mathbf{a}(B_2(2^n))$, where $\mathbf{a}(B_2(2^n))$ is taken from table 3.2.

Let $\dim S=2$. Then $R=S$ and, by lemma 3.1.10, $|A|\leqslant(q+1)^2<q^3=\mathbf{a}(B_2(2^n))$, where $\mathbf{a}(B_2(2^n))$ is taken from table 3.2.

§8 Large abelian subgroups in $G_2(q)$

By proof of lemma 3.1.9 it follows that A lies in the centralizer of some semisimple element. Since $G_2(K)$ is simply connected, a centralizer of any semisimple element is connected (see, for example, [21, theorem 2.11]). Therefore, A lies in a set of σ-stable points of some σ-stable connected reductive subgroup of maximal rank of $G_2(K)$. By Borel and deSiebental, there exist two maximal connected reductive subgroups R of maximal rank of $G_2(K)$; they are equal to $A_2(K)$ and $A_1(K)*A_1(K)$.

Let $R=A_2(K)$. Then, by lemma 3.1.1, A contains a subgroup A_0 of index 3 such that $A_0\leq A_2(q)$ or $A_0\leq {}^2A_2(q^2)$ (by [16, table 4 on page 138] both cases may occur). Note, that the bound of index from lemma 3.1.1 is rather bad (especially in case, when the number of factors is small), since we divide by some of numbers several times. In case, when $A_2(q)$ (or ${}^2A_2(q^2)$) is universal, we have, that it is coincide with R_σ, hence $A\leq A_2(q)$ (or $A\leq {}^2A_2(q^2)$). If $A_2(q)$ (or ${}^2A_2(q^2)$) is simple, then $A_0\leq A_2(q)$ (or $A_0\leq {}^2A_2(q^2)$), but $Z(A_2(q))$ (or $Z({}^2A_2(q^2))$) is trivial. By results for classical groups obtained above, we have that the order of A is not greater than either $3q^2$ or $(q+1)^2$. These numbers are clearly not greater than $\mathbf{a}(G_2(q))$, where $\mathbf{a}(G_2(q))$ is taken from table 3.2. Below we

refer to lemma 3.1.1 and omit arguments, but usually we use proof of the lemma instead of result.

Let $R = A_1(K) * A_1(K)$. Then by lemma 3.1.1 it follows that either $|A| \leqslant (2, q-1)q^2$, or $|A| \leqslant (q+1)^2$. In both cases $|A|$ is not greater, than $\mathbf{a}(G_2(q))$, where $\mathbf{a}(G_2(q))$ is taken from table 3.2.

§9 Large abelian subgroups in $F_4(q)$ and ${}^2F_4(q)$

Since $F_4(K)$ is simply connected, a centralizer of any semisimple element is connected, so A is contained in some proper maximal connected reductive subgroup of maximal rank of $F_4(K)$. Consider firstly $F_4(q)$.

By Borel and deSiebental algorithm, maximal proper connected σ-stable reductive subgroups of maximal rank of $F_4(K)$ may be found in the following list $B_4(K)$, $A_2(K) * A_2(K)$, $A_1(K) * A_3(K)$, and $C_3(K) * A_1(K)$.

Let A lies in $R = B_4(K)$. Therefore, by lemma 3.1.1 $|R_\sigma : O^{p'}(R_\sigma)| \leqslant 2$. Thus A contains a subgroup A_0 of index at most 2 such that $A_0 \leq B_4(q)$. Since the number $\mathbf{a}(B_4(q))$ is already found, we have that $|A| \leqslant 2q^7$ if q is odd, and $|A| \leqslant q^{10}$ if q is even. In both cases the order of A is not greater than $\mathbf{a}(F_4(q))$.

Assume, that A lies in $R = A_1(K) * C_3(K)$. Then by lemma 3.1.1 it follows that $|A| \leqslant 4q^7$. So the order of A is less than $\mathbf{a}(F_4(q))$.

Suppose A lies in $R = A_1(K) * A_3(K)$. Then $O^{p'}(R_\sigma)$ is isomorphic to one of the following group: $A_1(q) * A_3(q)$, or $A_1(q) * {}^2A_3(q^2)$ (by [16, table 2 on page 133] it follows that both cases may occur). By using lemma 3.1.1 and known result for classical groups we obtain that $|A| \leqslant 8q^5$. Thus $|A| \leqslant \mathbf{a}(F_4(q))$.

Let A is contained in $R = A_2(K) * A_2(K)$. Then $O^{p'}(R_\sigma)$ is isomorphic to either $A_2(q) * A_2(q)$, or ${}^2A_2(q^2) * {}^2A_2(q^2)$ (we again use [16, table 2 on page 133]). It follows that $|A| \leqslant 9q^4 \leqslant \mathbf{a}(F_4(q))$.

Consider ${}^2F_4(q)$. A is contained in some connected reductive subgroup R of maximal rank of $F_4(K)$. More over R is invariant under graph automorphism. Then by [16, table 3 on page 137] it follows, that either $A \leq {}^2B_2(q) * S$, or $A \leq {}^2A_2(q^4)$. So $|A|$ is not greater than $\mathbf{a}({}^2F_4(q))$, where $\mathbf{a}({}^2F_4(q))$ is taken from table 3.2.

§10 Large abelian subgroups of $E_6(q)$

For any connected algebraic group G of adjoin type and for any semisimple element $s \in G$ the group $C_G(s)/C_G^0(s)$ is known to be isomorphic to $\Gamma_{sc}/\Gamma_\pi = \Omega_\pi$ (see [50, Iwahory lecture, proposition 5]). Since for the root system of type E_6 we have $|\Gamma_{sc} : \Gamma_\pi| = 3$, there exists subgroup A_0 of A of index at most 3 such that A_0 is contained in some proper connected σ-stable reductive subgroup of maximal rank of $E_6(K)$. Any maximal connected reductive subgroups of $E_6(K)$ may be found in the following list: $A_1(K) * A_5(K)$, $A_2(K) * A_2(K) * A_2(K)$, and $S * D_5(K)$.

Let A_0 lies in $R = A_1(K) * A_5(K)$. Then $O^{p'}(R_\sigma) = A_1(q) * A_5(q)$ (see [16, case $E_6(q)$]). Then, by lemma 3.1.1, it follows that $|A_0| \leqslant 12q^{10}$. Hence $|A| \leqslant 36q^{10} \leqslant \mathbf{a}(E_6(q))$,

Let A_0 lies in $R = A_2(K) * A_2(K) * A_2(K)$. Then $O^{p'}(R_\sigma)$ is isomorphic to one of the following group $A_2(q) * A_2(q) * A_2(q)$, $A_2(q^2) * {}^2A_2(q^2)$, or $A_2(q^3)$ (see [16, case $E_6(q)$]).

In all cases, by lemma 3.1.1 we obtain the following bound $|A_0| \leqslant 27q^9$ (if $q = 2$, then $|A_0| \leqslant 9 \cdot 2^6$). It follows that $|A|$ is less than $\mathbf{a}(E_6(q))$.

Assume, that A_0 lies in $R = S * D_5(K)$. Then $R_1 = (S_\sigma) * D_5(q)$ (R_1 is defined in lemma 3.1.1). By lemma 3.1.1 it follows that $|A_0| \leqslant 4(q-1)q^{10}$. Hence $|A|$ is less than $\mathbf{a}(E_6(q))$.

§11 Large abelian subgroups in ${}^2E_6(q^2)$

The index $|\Gamma_{sc} : \Gamma_{ad}| = 3$, so A contains a subgroup A_0 of index at most 3 such that A_0 is contained in some proper connected σ-stable reductive subgroup R pf maximal rank of $E_6(K)$. Any maximal σ-stable connected reductive subgroup of maximal rank of $E_6(K)$ is isomorphic to one of the following list: $A_1(K) * A_5(K)$, $S * D_5(K)$, and $A_2(K) * A_2(K) * A_2(K)$. If the characteristic of K is even, then there does not exist a semisimple element s of $E_6(K)$ such that $C_{E_6(K)}(s)^0 = A_1(K) * A_5(K)$. In this case we have to consider two additional connected reductive σ-stable subgroups of maximal rank of $E_6(K)$: $A_5(K) * S$ and $A_4(K) * A_1(K) * S$.

Let A_0 lies in $R = A_1(K) * A_5(K)$ (recall, that q is odd in this case). Then $O^{p'}(R_\sigma) = A_1(q) * {}^2A_5(q^2)$ (see [16, table 1 on page 127]). By lemma 3.1.1 we obtain that $|A| \leqslant 24q^{10}$. Hence $|A| \leqslant \mathbf{a}({}^2E_6(q^2))$ if $q > 3$. If $q = 3$ we note, that $E_6(K)$ may be taken simply connected (since groups ${}^2E_6(3^2)_{sc}$ and ${}^2E_6(3^2)_{ad}$ are isomorphic). Therefore A itself is contained in R (instead of its subgroup of index 3). More over, by lemma 3.1.1 we obtain that $|A| \leqslant 4q^{10} \leqslant \mathbf{a}({}^2E_6(q^2))$.

Let A_0 lies in $R = S * D_5(K)$. Then $O^{p'}(R_\sigma) = (S_\sigma) * {}^2D_5(q^2)$ (see [16, table 1 on page 127]). Therefore, $|A| \leqslant 12(q+1)q^8$ ($|A| \leqslant 3q^8$, if q is even), i. e. the order of A is not greater than $\mathbf{a}({}^2E_6(q^2))$.

Assume, that A_0 lies in $R = A_2(K) * A_2(K) * A_2(K)$. Then $O^{p'}(R_\sigma)$ coincide with one of the following group $A_2(q^2) * A_2(q)$, ${}^2A_2(q^6)$, or ${}^2A_2(q^2) * {}^2A_2(q^2) * {}^2A_2(q^2)$ (see [16, table 1 on page 127]). Anyway we have the following estimate $|A| \leqslant 27q^6 \leqslant \mathbf{a}({}^2E_6(q^2))$.

Let A_0 lies in $R = A_5(K) * S$ (this case we consider only when the q is even). Then R_1 (defined in lemma 3.1.1) coincide with ${}^2A_5(q^2) * S_\sigma$. It follows that the order of A is not greater than $6(q+1)q^9$ ($3q^9$ if $q = 2$), and it is less than $\mathbf{a}({}^2E_6(q^2))$.

Suppose, that A_0 lies in $R = A_4(K) * A_1(K) * S$ (q is again assumed to be even). Then R_1 coincide with ${}^2A_4(q^2) * A_1(q) * S_\sigma$, hence $|A| \leqslant 15(q+1)^2q^6$ ($|A| \leqslant 3^2 \cdot 2^6$ if $q = 2$). So the order of A is less than $\mathbf{a}({}^2E_6(q^2))$.

§12 Large abelian subgroups in $E_7(q)$

We have that $|\Gamma_{sc} : \Gamma_{ad}| = 2$, so A contains a subgroup A_0 of index at most 2 such that A_0 is contained in some proper connected reductive σ-stable subgroup R of maximal rank of $E_7(K)$. Any maximal connected reductive σ-stable subgroup of maximal rank of $E_7(K)$ is isomorphic to one of the following groups: $A_7(K)$, $A_1(K) * D_6(K)$, $A_1(K) * A_3(K) * A_3(K)$, $A_2(K) * A_5(K)$, $S * E_6(K)$.

Let A_0 lies in $R = A_7(K)$. Then either $O^{p'}(R_\sigma) = A_7(q)$, or $O^{p'}(R_\sigma) = {}^2A_7(q^2)$ (see [17, table 1]). By lemma 3.1.1 we have that $|A| \leqslant 16q^{16}$, i. e. the order of A is not greater than $\mathbf{a}(E_7(q))$.

Let A_0 lies in $R = A_1(K) * D_6(K)$. Then $O^{p'}(R_\sigma)$ coincide with $A_1(q) * D_6(q)$ (see [17, table 1]). Therefore, $|A| \leqslant 16q^{16} \leqslant \mathbf{a}(E_7(q))$.

Assume, that A_0 lies in $R = A_1(K) * A_3(K) * A_3(K)$. Then $O^{p'}(R_\sigma)$ is isomorphic to either $A_1(q) * A_3(q) * A_3(q)$, or $A_1(q) * A_3(q^2)$ (see [17, table 1]). Anyway, by lemma 3.1.1, it follows that $|A| \leqslant 64q^9 \leqslant \mathbf{a}(E_7(q))$.

Suppose A_0 lies in $A_2(K) * A_5(K)$. Then $O^{p'}(R_\sigma)$ is isomorphic to either $A_2(q) * A_5(q)$, or ${}^2A_2(q^2) * {}^2A_5(q^2)$ (see [17, table 1]). Therefore the following inequality holds $|A| \leqslant 36q^{11} < \mathbf{a}(E_7(q))$.

Suppose, that A lies in $R = S * E_6(K)$. Then R_1 is isomorphic to either $(S_\sigma) * E_6(q)$, or $(S_\sigma) * {}^2E_6(q^2)$. Anyway, $|A| \leqslant 6(q-1)q^{16} < \mathbf{a}(E_7(q))$.

§13 Large abelian subgroups in $E_8(q)$

Since $\Gamma_{sc} = \Gamma_{ad}$, a centralizer of any semisimple element of $E_8(K)$ is connected. Therefore, A is contained in some proper connected reductive σ-stable subgroup of maximal rank of $E_8(K)$. Any maximal connected reductive σ-stable subgroup of maximal rank of $E_8(K)$ is isomorphic to: $D_8(K)$, $A_8(K)$, $A_1(K) * A_2(K) * A_5(K)$, $A_4(K) * A_4(K)$, $A_3(K) * D_5(K)$, $A_2(K) * E_6(K)$, $A_1(K) * E_7(K)$.

Let A lies in $R = D_8(K)$. Then $O^{p'}(R_\sigma) = D_8(q)$ (see [17, table 2]). By lemma 3.1.1 it follows that $|A| \leqslant 4q^{27} < \mathbf{a}(E_8(q))$.

Let A lies in $R = A_8(K)$. Then either $O^{p'}(R_\sigma) = A_8(q)$, or $O^{p'}(R_\sigma) = {}^2A_8(q^2)$ (see [17, table 2]). By lemma 3.1.1 we obtain, that $|A| \leqslant 9q^{20} < \mathbf{a}(E_8(q))$.

Let A lies in $R = A_1(K) * A_2(K) * A_5(K)$. Then $O^{p'}(R_\sigma)$ is isomorphic to either $A_1(q) * A_2(q) * A_5(q)$, or $A_1(q) * {}^2A_2(q^2) * {}^2A_5(q^2)$ (see [17, table 2]). Again, by lemma 3.1.1, it follows, that $|A| \leqslant 36q^{12} < \mathbf{a}(E_8(q))$.

Let A lies in $R = A_4(K) * A_4(K)$. Then $O^{p'}(R_\sigma)$ is isomorphic to one of the following group: $A_4(q) * A_4(q)$, ${}^2A_4(q^2) * {}^2A_4(q^2)$, or ${}^2A_4(q^4)$ (see [17, table 2]). By lemma 3.1.1 it follows, that $|A| \leqslant 25q^{12} < \mathbf{a}(E_8(q))$.

Let A lies in $R = A_3(K) * D_5(K)$. Then $O^{p'}(R_\sigma)$ is isomorphic to either $A_3(q) * D_5(q)$, or ${}^2A_3(q^2) * {}^2D_5(q^2)$ (see [17, table 2]). By lemma 3.1.1, we obtain, that $|A| \leqslant 16q^{19} < \mathbf{a}(E_8(q))$.

Suppose A lies in $R = A_2(K) * E_6(K)$. Then $O^{p'}(R_\sigma)$ is isomorphic to either $A_2(q) * E_6(q)$, or ${}^2A_2(q^2) * {}^2E_6(q^2)$. So $|A| \leqslant 9q^{18} < \mathbf{a}(E_8(q))$.

Assume, at least, that A lies in $R = A_1(K) * E_7(K)$. Then $O^{p'}(R_\sigma) = A_1(q) * E_7(q)$. By lemma 3.1.1, $|A| \leqslant 4q^{28} < \mathbf{a}(E_8(q))$.

§14 Large abelian subgroups in ${}^3D_4(q^3)$

There exist two maximal connected σ-stable subgroups of maximal rank of $D_4(K)$: $A_1(K) * A_1(K) * A_1(K) * A_1(K)$ and $T * A_2(K)$, where T is a torus of dimension 2.

Let A lies in $R = A_1(K) * A_1(K) * A_1(K) * A_1(K)$. Then $O^{p'}(R_\sigma) = A_1(q) * A_1(q^3)$ (see [16, table 7 on page 140]). By lemma 3.1.1 it follows, that $|A| \leqslant 4q^4$ ($|A| \leqslant 27$ if $q = 2$ and $|A| \leqslant 2q^4$, if $q = 3$). So $|A| \leqslant q^5 = \mathbf{a}({}^3D_4(q^3))$.

Assume, that A lies in $R = T * A_2(K)$. Then either $O^{p'}(R_\sigma) = (T_\sigma) * A_2(q)$, or $O^{p'}(R_\sigma) = (T_\sigma) * {}^2A_2(q^2)$ (see [16, table 7 on page 140]). By lemma 3.1.1 it follows, that

$|A| \leqslant 3(q^2+q+1)q^2$ ($|A| \leqslant 28$ if $q=2$, and $|A| \leqslant 117$ if $q=3$). Thus $|A| \leqslant q^5 = \mathbf{a}({}^3D_4(q^3))$.

§15 Summary table

In table 3.2 below we give orders of large abelian subgroups in all finite groups of Lie type. In the column "source" we give the reference to papers, where the reader may find the structure of subgroups of given order.

Table 3.2. Large abelian subgroups of finite groups of Lie type

group G	$\mathbf{a}(G)$	source	group G	$\mathbf{a}(G)$	source
$A_n(q)$ except $A_1(q)$, $q=2^\alpha$ and $A_2(q)$, $(3,q-1)=1$	$q^{[(n+1)^2/4]}$	[3]	${}^2D_n(q^2)$, $n \geqslant 5$	$q^{(n-1)(n-2)/2+2}$	[45]
			${}^2D_4(q^2)$	q^6	[45]
			$G_2(q)$, $q \neq 3^\alpha$ except $G_2(2)$	q^3	[61]
$A_1(q)$, $q=2^\alpha$	$q+1$	[59]			
$A_2(q)$, $(3,q-1)=1$	q^2+q+1	[59]	$G_2(q)$, $q=3^\alpha$	q^4	[61]
$B_n(q)$, $n \geqslant 4$, $q \neq 2^\alpha$	$q^{n(n-1)/2+1}$	[3], [45]	$F_4(q)$, $q=2^\alpha$	q^{11}	[63]
$B_3(q)$, $q \neq 2^\alpha$	q^5	[3], [45]	$F_4(q)$, $q \neq 2^\alpha$	q^9	[63]
$C_n(q)$, except $C_2(2)$	$q^{n(n+1)/2}$	[3], [4]	$E_6(q)$,	q^{16}	[63]
$D_n(q)$	$q^{n(n-1)/2}$	[3], [45]	$E_7(q)$,	q^{27}	[63]
${}^2A_n(q^2)$, except ${}^2A_2(q^2)$, $(3,q+1)=1$ and ${}^2A_3(2^2)$	$q^{[(n+1)^2/4]}$	[46]	$E_8(q)$,	q^{36}	[63]
			${}^2B_2(q)$	$2q$	[40]
			${}^2G_2(q)$	q^2	[41]
${}^2A_2(q^2)$, $(3,q+1)=1$	$(q+1)^2$	[59]	${}^3D_4(q^3)$	q^5	[61]
${}^2A_3(2^2)$	27	[59]	${}^2E_6(q^2)$	q^{12}	[63]
			${}^2F_4(q)$	$2q^5$	[63]

Chapter 4

Large nilpotent subgroups of finite simple groups

§1 Large nilpotent subgroups of symmetric and alternating groups

First we prove the following technical lemma.

Lemma 4.1.1. *Let N be a nilpotent subgroup of S_n and $I_1, I_2, \ldots$ be a set of orbits of the center $Z(N)$ in N on a set $\{1, \ldots, n\}$. Assume that J_1 is a collection of sets I_m of order 1, J_2 is a collection of I_m of order 2, etc. Suppose that $K_1 = \bigcup_{|I_m|=1} I_m$, $K_2 = \bigcup_{|I_m|=2} I_m$, etc. Then the following statements hold:*

1. *the group $N/Z(N)$ permutes sets of one order, and consequently $N \leq N_1 \times N_2 \times \ldots$, where $N_1 \leq S_{K_1}$, $N_2 \leq S_{K_2}$, etc.;*
2. *if k_i is the number of orbits under the action of $N/Z(N)$ on J_i then $|Z(N) \cap N_i| = i^{k_i}$;*
3. *if $p_1, \ldots, p_s$ are all primes by which i is divisible then the order of N_i is divisible only by $p_1, \ldots, p_s$.*

Proof. 1. Let σ be an element of the group N which translates the element i of the set I_1 into an element of some set I_k. Then $i^{\sigma\tau} = i^{\tau\sigma} \in I_k$ for any $\tau \in Z(N)$. Since $Z(N)$ acts transitively on I_1, we have $\{i^\tau : \tau \in Z(N)\} = I_1$; consequently, $I_1^\sigma \subseteq I_k$, that is, $|I_1| \leqslant |I_k|$. On the other hand, the element σ^{-1} translates an element of I_k into the element i of I_1; therefore, $I_k^{\sigma^{-1}} \subseteq I_1$, that is, $|I_k| \leqslant |I_1|$. Combining the inequalities obtained yields $|I_1| = |I_k|$.

2. We may assume that $K_i = \{1, \ldots, n\}$ and the group $N/Z(N)$ acts transitively on the orbits (under the action of the center $Z(N)$) of J_i. Let $\{I_1, \ldots, I_k\}$ be a set of all orbits of J_i under the action of $Z(N)$. Then the order of each such orbit equals i, and $i \cdot k = n$. Let $l \in I_1$ be some element of the orbit I_1. Consider a stabilizer $St_{Z(N)}(l)$ of an element l in the center $Z(N)$ and assume that $\tau \in St_{Z(N)}(l)$. Let $m \in K_i$ be an element lying in some orbit I_j. Since $N/Z(N)$ acts transitively on $I_1, \ldots, I_k$, there exists an element $\sigma \in N$ such that $I_j^\sigma = I_1$. Further, the group $Z(N)$ acts transitively on I_1, and so there exists an element $\varphi \in Z(N)$ such that $(m^\sigma)^\varphi = l$. It follows that

$m^\tau = ((l^{\varphi^{-1}})^{\sigma^{-1}})^\tau = ((l^\tau)^{\varphi^{-1}})^{\sigma^{-1}} = m$; consequently, $\tau = \varepsilon$ and $St_{Z(N)}(l) = \{\varepsilon\}$. By the Lagrange theorem, $|Z(N)| = |Z(N) : St_{Z(N)}(l)| \cdot |St_{Z(N)}(l)| = i$.

3. Further, $J_i = \{I_k : |I_k| = i\}$ by definition. Assume that there exists a prime $q \notin \{p_1, \dots, p_s\}$ which divides the order of N_i. Since N_i is a nilpotent group, there exists a central element τ of order q. Because N is a direct product of groups $N_1, N_2, \dots$, the element τ lies in $Z(N)$. Clause (2) implies that $|Z(N) \cap N_i| = i^k$, where $k \geqslant 1$. Hence τ lies in $Z(N) \cap N_i$, but $|\tau|$ does not divide $|Z(N) \cap N_i|$, a contradiction. ⊠

Note that the group N_1 specified in the lemma is trivial. Now we are in a position to explicate the structure of symmetric and alternating groups.

Theorem 4.1.2. *A large nilpotent subgroup in an alternating group is conjugate to one of the following groups:*

1. $\langle(1,2,3)\rangle$ *if* $n = 3$*;*
2. $\langle(1,2,3,4,5)\rangle$ *if* $n = 5$*;*
3. $\langle(1,2,3)\rangle \times \langle(4,5,6)\rangle$ *if* $n = 6$*;*
4. $Syl_2(A_n)$ *if* $n \neq 2(2k+1)+1$ *for some natural* k*;*
5. $Syl_2(A_{n-3}) \times \langle(n-2, n-1, n)\rangle$ *if* $n = 2(2k+1)+1$*,* $k \geqslant 1$*.*

A large nilpotent subgroup in a symmetric group is conjugate to one of the following:

1. $Syl_2(S_n)$ *if* $n \neq 2(2k+1)+1$ *for some natural* k*;*
2. $Syl_2(S_{n-3}) \times \langle(n-2, n-1, n)\rangle$ *if* $n = 2(2k+1)+1$ *for some natural* k*.*

In all groups, a large nilpotent subgroup is unique up to conjugation.

Proof. Assume that the statement of the theorem fails and that n is the minimal natural number yielding a counterexample to the hypothesis. Let P be a subgroup of S_n which is structured the same way as is the nilpotent subgroup specified in the theorem. Suppose that $N \in N(S_n)$ is a large nilpotent subgroup which is not conjugate to p.

Under the action of $Z(N)$, the set $\{1, \dots, n\}$ gets partitioned into orbits. There are two cases to consider:

1. Among the orbits of the center $Z(N)$, there are subsets of different orders. By Lemma 4.1.1, therefore, $N(S_n)$ is a subgroup in the direct product of the groups $N_1 \leq S_{n_1}$ and $N_2 \leq S_{n_2}$, in which case $n_1 + n_2 = n$. Since n is the minimal natural number for which the statement of the theorem fails, the groups of $N(S_{n_1})$ and $N(S_{n_2})$ are structured in the way specified by the theorem.

Let $n_1 \neq 2(2k+1)+1$; then $|N_1| \leqslant |S|$, where $S \in Syl_2(S_{n_1})$. Depending on whether or not the number n_2 is representable as $2(2k+1)+1$, we obtain the following values: $|N_2| \leqslant 3|S_1|$ or $|N_2| \leqslant |S_2|$, where $S_1 \in Syl_2(S_{n_2-3})$ and $S_2 \in Syl_2(S_{n_2})$. For the first option, we have $|N| \leqslant |N_1| \cdot |N_2| \leqslant |Syl_2(S_{n_1})| \cdot 3|Syl_2(S_{n_2-3})| \leqslant 3|Syl_2(S_{n-3})| \leqslant |P|$, in which case the equality attains only if $N_1 \in Syl_2(S_{n_1})$ and $N_2 = S_1 \times \langle(k_1, k_2, k_3)\rangle$, that is, if $N = P$ up to conjugation. Which is impossible, for n is the minimal number delivering a counterexample. Similarly we can treat the situation where $N_2 \in Syl_2(S_{n_2})$.

Let $n_1 = 2(2k_1 + 1) + 1$ and $n_2 = 2(2k_2 + 1) + 1$. Then $|N(G)| \leqslant 3|S_3| \cdot 3|S_1| < |Syl_2(S_n)| = |P|$, where $S_3 \in Syl_2(S_{n_1-3})$, and we arrive at a contradiction. Thus the first case which holds that orbits may contain subset of different orders is impossible.

2. Suppose that all orbits under the action of $Z(N)$ are of the same order k. Let $I_1, \ldots, I_{n/k}$ all be orbits of the set $\{1, \ldots, n\}$ under the action of $Z(N)$. If the action of the group $N/Z(N)$ on a set of orbits $I_1, \ldots, I_{n/k}$ is not transitive, N is a subgroup in the direct product of N_1 and N_2, each of which is a nilpotent subgroup of a symmetric group of lesser degree. Similarly to the first case, we can show that N is not a counterexample.

Now we let N act transitively on the orbits $I_1, \ldots, I_{n/k}$. In virtue of lemma 4.1.1(2), the order of $Z(N)$ equals k, the group $N/Z(N)$ can be treated as a nilpotent subgroup of $S_{n/k}$, and so $|N| \leqslant k \cdot |N_3|$, where $N_3 \in N(S_{n/k})$. It is not hard to verify that $|N| \leqslant |Syl_2(S_n)|$ except $n = k = 3$. We have thus proved the theorem for symmetric groups.

We turn to alternating groups. Let n be the minimal number for which a counterexample to the statement of the theorem exists. Let that counterexample be furnished by $N \in N(A_n)$. Write R to denote a nilpotent subgroup of A_n which coincides with the large nilpotent group specified by the theorem. Again we have two cases to consider:

1. The action of $N/Z(N)$ on a set of orbits of the center $Z(N)$ is not transitive. Hence, either N is contained in a direct product of nilpotent groups N_1 and N_2 each of which is a nilpotent subgroup of an alternating group in a lesser dimension, or it belongs to a direct product of two nilpotent groups N_1 and N_2, of which each is a nilpotent subgroup of a symmetric group in a lesser dimension, but does not coincide with that product. Using the orders of large nilpotent subgroups in symmetric groups at hand, it is not hard to verify that N fails as a counterexample in this case, too.

2. The group $N/Z(N)$ acts transitively on a set of orbits. Suppose that each orbit is of order k. By lemma 4.1.1, then, $|Z(N)| = k$ and $N/Z(N)$ can be treated as a nilpotent subgroup of $S_{n/k}$. It is not hard to verify that $k|N_4| \leqslant (1/2)|S| \leqslant |R|$, where $N_4 \in N(A_{n/k})$ and $S \in Syl_2(S_n)$, holds for $n \geqslant 7$. ⊠

Table 4.1. Nilpotent subgroups of maximal order in symmetric and alternating groups

Group G	$\mathbf{n}(G)$	Structure
A_3	3	$\langle(1,2,3)\rangle$
A_5	5	$\langle(1,2,3,4,5)\rangle$
A_6	9	$\langle(1,2,3)\rangle \times \langle(4,5,6)\rangle$
A_n, $n \neq 2(2k+1)+1$	$\frac{1}{2}2^{[n/2]+[n/2^2]+\ldots}$	S, where $S \in \mathrm{Syl}_2(A_n)$
A_n, $n = 2(2k+1)+1$	$\frac{3}{2}2^{[(n-3)/2]+[(n-3)/2^2]+\ldots}$	$S \times \langle(n-2, n-1, n)\rangle$, $S \in \mathrm{Syl}_2(A_{n-3})$
S_n, $n \neq 2(k+1)+1$	$2^{[n/2]+[n/2^2]+\ldots}$	S, $S \in \mathrm{Syl}_2(S_n)$
S_n, $n = 2(2k+1)+1$	$3 \cdot 2^{[(n-3)/2]+[(n-3)/2^2]+\ldots}$	$S \times \langle(n-2, n-1, n)\rangle$, $S \in \mathrm{Syl}_2(S_{n-3})$

§2 General structure of nilpotent subgroups in simple algebraic groups

Lemma 4.2.1. *Let N be a closed nilpotent subgroup of a connected, simple, algebraic group G. Then there exists a reductive subgroup R of maximal rank in G, containing a group N. Let W_1 be a Weyl group of R^0. Then the following statements hold:*

1. *$N = N_s \times N_u$, that is, N is representable as a direct product of its subgroups consisting of semisimple and unipotent elements;*
2. *$N_u \leq R^0$ and $Z(N_s) \cap R^0 \leq Z(R^0)$;*
3. *if $N_0 = N \cap R^0$ then N/N_0 is isomorphically embeddable in the group $N_W(W_1)/W_1$.*

If N consists of σ-invariant elements under some Frobenius automorphism σ, then the group R is σ-invariant.

Proof. Let N be a closed nilpotent subgroup of a connected, simple, algebraic group defined over an algebraic closure of a field $GF(q)$. Then N consists of elements of finite orders and is representable as a direct product of its p-subgroups (cf. [37, 12.1.1]). In particular, N can be represented as $N_s \times N_u$, which is a direct product of its semisimple and unipotent parts, respectively.

If the group N_s is non-trivial then its center is also. Clearly, $Z(N_s) = (Z(N))_s$. Let x be some element of $Z(N_s)$. Then $N \subseteq C_G(x)$, with $N_u \subseteq R^0$. Denote by R the group $C_G(x)$. By Lemma 3.1.2, R is a reductive subgroup of maximal rank in G. Suppose that there exists an element s of $Z(N_s) \cap R^0$ which does not lie in $Z(R^0)$. Consider a group $C_R(s)$. Clearly, $N \leq C_R(s)$ and $N_u \leq R^0$. Again, $C_R(s)$ is a reductive subgroup of maximal rank in G. Since R decreases in dimension at each step, the process is finite (the dimension of G is finite). Allowing a repetition of the above process yields a reductive subgroup R of maximal rank in G containing N. If N consists of fixed points w.r.t. some Frobenius automorphism σ, R will be σ-invariant. We have thus proved clauses (1) and (2) of the lemma.

We turn to (3). We have $N/N_0 = NR^0/N_0R^0 \leq R/R^0$. The proof of lemma 3.1.2 implies that every element of R is representable as nx, where $n \in N_R(T)$ for some maximal torus T of R^0, and $x \in R^0$. Since R^0 is normal in R, the group $N_R(T)/T$ is contained in the group $N_W(W_1)$. This gives us $R/R^0 \cong N_R(T)/N_{R^0}(T) \leq N_W(W_1)/W_1$. ⊠

Remark. Lemma 4.2.1 implies that $N_0/Z(N_0)$ is a nilpotent subgroup in a direct product of simple algebraic groups of lesser dimension — the group $R^0/Z(R^0)$. The lemma thus generalizes a result of [1] concerning the structure of semisimple nilpotent subgroups in generalized linear groups over finite fields, since for $GL_n(K)$ the equality $N_0 = N$ holds.

We know how reductive subgroups R of maximal rank in G, and also subgroups R_σ, are structured; see [9], [10], [16], and [17]. To treat nilpotent subgroups of finite groups of Lie type, therefore, we are left to find orders of large nilpotent subgroups in Weyl groups. The Weyl groups for types B_n, C_n, and D_n are a wreath product of a 2-group and a symmetric group S_n. The data obtained on the structure of nilpotent subgroups in symmetric groups can now be used to conclude that a large nilpotent subgroup in a Weyl group for all the types mentioned is exactly a Sylow 2-group. Table 4.2 shows values

for orders of large nilpotent subgroups in Weyl groups for all classical groups, and their structure.

Table 4.2. Nilpotent subgroups of maximal orders in Weil groups

Type of system Φ	Structure of groups $\mathbf{N}(W(\Phi))$	Bound for $\mathbf{n}(W(\Phi))$
A_n	see table 4.1	2^{n+1}
B_n and C_n	in $\mathrm{Syl}_2(W)$	2^{2n}
D_n	in $\mathrm{Syl}_2(W)$	2^{2n-1}

§3 Large nilpotent subgroups of finite groups of lie type

Here, we work to apply the above-specified general properties of nilpotent subgroups to finite Lie-type groups. In particular, we prove that a large nilpotent subgroup coincides, in most of the cases, with a maximal unipotent subgroup. Finding large nilpotent subgroups in finite Chevalley groups proceeds uniformly, so we treat $A_n(q)$ to exemplify this process.

Let N be some nilpotent subgroup of $A_n(q)$. We claim that its order does not exceed the order of the large nilpotent group indicated in table 4.3. We may assume that the center of $A_n(q)$ is trivial. By lemma 4.2.1, then, the group N is contained in some proper reductive subgroup of maximal rank in a connected, simple, algebraic group of type A_n.

First we recall the structure of reductive subgroups of maximal rank in a simple, connected, algebraic group of type A_n, and also how are structured their fixed points under the Frobenius automorphism σ; see [9]. Assume that G is of type A_n. The endomorphism σ of G induces an endomorphism of the character group X of a torus T, which is also denoted by σ and has the form $\sigma = q\sigma_0$, where q is the power of p and σ_0 is an isometry of X. The isometry σ_0 has order 1 or 2, depending on whether G_σ is normal or twisted. The group X contains the set Φ of roots, and Φ is conveniently represented as $\Phi = \{e_i - e_j : i \neq j, i, j \in \{0, 1 \ldots, n\}\}$, where $e_0, e_1, \ldots, e_n$ form an orthonormal basis for an $(n+1)$-dimensional Euclidean space. The Weyl group W acts on that space by permuting the basis elements in a way that fits the symmetric group S_{n+1}. The isometry σ_0 acts on the roots either identically or as an element of order 2, mapping e_i into e_{n-i}.

The root system of any σ-invariant reductive subgroup of G is equivalent w.r.t. W to a system Φ_1 of the following type. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be the partition of $n+1$ and $I_1, I_2, \ldots$ be disjoint subsets of $\{0, 1, \ldots, n\}$ satisfying the condition that $|I_1| = \lambda_1$, $|I_2| = \lambda_2$, ... Assume $\Phi_1 = \{e_i - e_j \in \Phi : i, j \in I_\alpha \text{ for some } \alpha\}$. Then Φ_1 is a subsystem of Φ of type $A_{\lambda_1 - 1} \times A_{\lambda_2 - 1} \times \ldots$. And it is σ-invariant on the condition that if σ_0 has order 2 then Φ_1 is invariant under a linear transformation given by the rule $e_i \to -e_{n-i}$.

Lemma 4.3.1. *[9, proposition 7]. Let G be a group of type A_l and let σ be such that G_σ is split. Let G_1 be a reductive subgroup of maximal rank in G corresponding to a partition λ of $l+1$. Let G_1^g be a σ-stable subgroup of G obtained by twisting G_1 by an*

element $w \in W$ defined by $\pi(g^\sigma g^{-1}) = w$. Suppose w maps to τ under the homomorphism $N_W(W_1) \to Aut_W(\Delta_1)$. Let m_i be the number of pairs of λ equal to i, so that $Aut_W(\Delta_1) \cong S_{m_2} \times S_{m_3} \times \ldots$. Suppose τ gives rise to partitions $\mu^{(2)}, \mu^{(3)}, \ldots$ of $m_2, m_3, \ldots$ respectively. Then the simple components of the group $(M^g)_\sigma$ are of type $A_{i-1}(q^{\mu_j^{(i)}})$ with one component for each $i = 2, 3, \ldots$ and each part $\mu_j^{(i)}$ of $\mu^{(i)}$.

The order of the semisimple part $(S^g)_\sigma$ of $(G_1^g)_\sigma$ is given by

$$(q-1)|(S^g)_\sigma| = \prod_{i,j}(q^{\mu_j^{(i)}} - 1).$$

Since the center of $A_n(q)$ is assumed trivial, the order of the centralizer specified in lemma 4.3.1 should be multiplied by $1/(n+1, q-1)$.

By lemma 4.3.1, there exists a subgroup N_0 of N lying in some σ-invariant, connected, reductive subgroup R of maximal rank in G. Furthermore, $|N : N_0| \leqslant 2^{n+1}$; see table 4.2. Since the center of $A_n(q)$ is assumed trivial, the group R is a proper subgroup of G. In this way N_0 is representable as a central product of nilpotent subgroups of groups in a lesser dimension and the group that is a fixed-point subgroup of some torus. Therefore, the order of N is estimated thus:

$$(q-1)|N| \leqslant \mathbf{n}(S_{n+1})\frac{1}{(n+1, q-1)}\prod_{i,j}(q^{\mu_j^{(i)}} - 1)\prod_{i,j}(i, q^{\mu_j^{(i)}} - 1)\mathbf{n}(A_{i-1}(q^{\mu_j^{(i)}})). \quad (4.1)$$

Here, as $A_{i-1}(q^{\mu_j^{(i)}})$ we consider a group with trivial center. Using induction on the Lie rank of a group, we can prove that the following hold:

$$(q^k - 1)(i, q^k - 1)\mathbf{n}(A_{i-1}(q^k)) \leqslant (q-1)(ik, q-1)\mathbf{n}(A_{ik-1}(q)), \quad (4.2)$$

$$(q-1)(i, q-1)\mathbf{n}(A_{i-1}(q))(q-1)(k, q-1)\mathbf{n}(A_{k-1}(q)) \leqslant (q-1)(ik, q-1)\mathbf{n}(A_{ik-1}(q) \quad (4.3)$$

(the number $\mathbf{n}(A_k(q))$ is taken from the table 4.3).

Using (4.2) and (4.3), the right part of (4.1) can be written either in the form

$$(q-1)^2\mathbf{n}(S_{n+1})(n_1, q-1)\mathbf{n}(A_{n_1-1}(q))(n_2, q-1)\mathbf{n}(A_{n_2-1}(q)), \quad (4.4)$$

where $n_1 + n_2 = n + 1$, or in the form

$$(q^2 - 1)\mathbf{n}(S_{n+1})((n+1)/2, q^2 - 1)\mathbf{n}(A_{(n+1)/2-1}(q^2)). \quad (4.5)$$

It is not hard to verify that (4.4) and (4.5) do not exceed the values indicated in table 4.3. Other finite groups of Lie type can be treated in a similar way.

Table 4.3 exhibits the structure of large unipotent subgroups for the case where a finite group G of a given type has trivial center. For groups with an arbitrary center, a large nilpotent subgroup is the preimage of a large nilpotent subgroup in the group with trivial center under the natural homomorphism.

Table 4.3. Nilpotent subgroups of maximal orders in finite groups of Lie type

Group G	Structure of groups in $\mathbf{N}(G)$	$\mathbf{n}(G)$
$A_1(2^n)$	cyclic group	2^n+1
$A_1(q)$, $q-1=2^n$	lies in $\mathrm{Syl}_2(A_1(q))$	2^n
${}^2A_2(2^2)$	lies in $\mathrm{Syl}_3({}^2A_2(2^2))$	27
${}^2A_2(3^2)$	lies in$\mathrm{Syl}_2({}^2A_2(3^2))$	32
for all other G	maximal unipotent subgroup	

§4 Large nilpotent subgroups of sporadic groups

In dealing with large nilpotent subgroups, we make use of the information in [15]. For all sporadic groups, a large nilpotent subgroup is a Sylow subgroup, and so finding large nilpotent groups calls for a uniform argument. We just outline the idea.

If N is a nilpotent subgroup of G, and $p_1, \ldots, p_k$ are all primes dividing the order of N, then N contains a central element of order $p_1 \cdot \ldots \cdot p_k$. The study of orders of centralizers of such elements using [15] shows that the order of N, in this case, is less than the order of a Sylow subgroup. An easy consequence of this is the following theorem 1.1.3.

Proof. If $N(G)$ coincides with $Syl_p(G)$ for some prime p, the statement of the theorem follows from [34, 2]. If $G = A_n$, $n = 2(2k+1)+1$ for some natural k, it is easy to see that $N(G)^2 < 2^{2(n-1)} < |G|$. Finally, if G coincides with $A_1(2^n)$, then the group $N(G)$ is Abelian, and by [14], $N(G)^2 < |G|$. ⊠

Chapter 5

Large normal nilpotent subgroups of finite soluble groups

§1 Available results

Lemma 5.1.1. *[37, theorem 5.3.3] If G is a group of order p^m and $|G : \Phi(G)| = p^r$ then the order of $C_{\mathrm{Aut}\ (G)}(G/\Phi(G))$ divides $p^{(m-r)r}$.*

Corollary 5.1.2. *Let G be a finite p-group. If the order of an automorphism α of G does not divide p and α acts trivially on $G/\Phi(G)$ then α centralizes G.*

Proof. Let α be a nontrivial automorphism whose order does not divide p. Then by lemma 5.1.1 α does not belong to $C_{\mathrm{Aut}\ (G)}(G/\Phi(G))$. ⊠

Lemma 5.1.3. *[37, theorem 5.3.2] If G is a finite p-group then $\Phi(G) = [G, G]G^p$.*

Corollary 5.1.4. *If G is a finite p-group then $G/\Phi(G)$ is an elementary abelian group.*

Proof. Since $[G, G] \leq \Phi(G)$, the group $G/\Phi(G)$ is abelian. Moreover, each element g of G raised to the power p belongs to $\Phi(G)$. ⊠

Lemma 5.1.5. *[37, theorem 5.2.4] If G is a finite group then the following properties are equivalent:*

(i) the group G is nilpotent;

(ii) the group G is a direct product of its Sylow subgroups.

Corollary 5.1.6. *Let G be a finite group and B a normal p-subgroup of G. Suppose that G contains an element α whose order does not divide p and which does not centralize B. Then G is not nilpotent.*

Proof. The group $\langle \alpha, B \rangle$ cannot be represented as a direct product of its Sylow subgroups; therefore, it is not nilpotent. In consequence, the whole group G is not nilpotent. ⊠

Lemma 5.1.7. *[37, theorem 5.4.4] If G is a solvable group and F is the Fitting subgroup of G then $C_G(F) = \zeta(F)$.*

Lemma 5.1.8. *[44, theorem 1.6] If G is a nilpotent subgroup of $GL(V)$ whose order is coprime to the characteristic of the field over which the finite vector space V is defined, then $|G| \leqslant |V|^{\beta}/2$, where $\beta = \log 32/\log 9$.*

Lemma 5.1.9. *[36]. The inequality $i_p(G) \leqslant 3$ holds for every finite solvable group G..*

Corollary 5.1.10. *Let P be a Sylow p-subgroup of a finite nontrivial solvable group G and $O_p(G) = \{e\}$. Then $|G : P|^2 > |P|$.*

PROOF. In view of Lemma 5.1.9, there are three Sylow p-subgroups P_1, P_2, and P_3 such that $P_1 \cap P_2 \cap P_3 = \{e\}$. Since G is not a p-group, the inequality $|G| > \frac{|P_1|\cdot|P_2|}{|P_1 \cap P_2|}$ holds. Furthermore, since $P_1 \cap P_2 \cap P_3 = \{e\}$, we have $|P_1 \cap P_2| \cdot |P_3| < |G|$. Thus,

$$|G| > \frac{|P_1| \cdot |P_2|}{|P_1 \cap P_2|} > \frac{|P_1| \cdot |P_2 \cdot |P_3| \cdot |P_1 \cap P_2|}{|G| \cdot |P_1 \cap P_2|} = \frac{|P_1|^3}{|G|}.$$

Therefore, $|G|^2 > |P_1|^3$, and $|G : P_1|^2 > |P_1|$ by Lagrange theorem [37, theorem 1.3.6]. ⊠

Remark. The claim of Lemma 6 is proved in [48] for all finite groups. However, the proof of this fact in the general case essentially uses the classification of finite simple groups.

§2 Proof of theorem 1.1.4

In this section theorem 1.1.4 we have stated in the Introduction

Let H be a nilpotent subgroup of G such that $|G : H| = n$. Consider $N = F(G)$. In view of lemma 5.1.5, $N = P_1 \times \cdots \times P_k$ where P_i are Sylow p_i-subgroups of N.

Consider the homomorphism $\varphi : G \to G/\Phi(N) = \overline{G}$, henceforth denoting the images of elements and sets under this homomorphism by overlining the letters signifying them.

Lagrange theorem implies $|\overline{G} : \overline{N}| = |G : N|$ and $|\overline{G} : \overline{H}| \leqslant |G : H|$; therefore, to prove theorem 1.1.4, it suffices to show that $|\overline{G} : \overline{N}| < |\overline{G} : \overline{H}|^5$.

By corollary 5.1.4, the group $\overline{N}$ can be represented as $\overline{N} = \overline{P_1} \times \cdots \times \overline{P_k}$, where $|\overline{P_i}| = p_i^{n_i}$ and $\exp(\overline{P_i}) = p_i$, i.e., as a direct product of elementary abelian groups. Thus, each $\overline{P_i}$ may be regarded as a vector space of dimension n_i over the field F_{p_i}. Since $N \trianglelefteq G$, we may consider the homomorphisms $\varphi_i : G \to GL(n_i, p_i)$, $i = 1, \ldots, k$. These homomorphisms induce homomorphisms $\varphi_i : \overline{G} \to GL(n_i, p_i)$ and $\varphi_i : G/N \to GL(n_i, p_i)$ which we denote by the same letters to simplify notation. Let N_1 be a subgroup of N invariant under conjugation by some element x of G. The element x acts *unipotently on* N_1 if for every i the image of x under φ_i acts unipotently on $\overline{N_1} \cap \overline{P_i}$. If we take as N_1 the whole group N then we say that x acts *unipotently.* By analogy we define the notion of unipotent action on the subgroup $\overline{N_1}$ for the elements $\overline{x} \in \overline{G}$ and $x \in G/N$. A subgroup U of G acts unipotently on a subgroup N_1 of N if each element of U acts unipotently on N_1 (surely, N_1 is assumed invariant under conjugation by the group U). In the case when N_1 coincides with N, we say that the group U acts unipotently. We define unipotent action for subgroups of the groups $\overline{G}$ and G/N in the same way as for elements.

With the above notations, the following holds:

Lemma 5.2.1. *Let U be a normal subgroup of G which acts unipotently. Then $U \leq N$, and so $\overline{U} \leq \overline{N}$ and G/N lacks nontrivial normal subgroups that act unipotently.*

PROOF. We may assume that $N \leq U$: otherwise the group NU is normal in G and acts unipotently.

Suppose that $U \neq N$ and V/N is a minimal characteristic subgroup of U/N. Then $V \trianglelefteq G$ and V/N is a p-group. Let P be a Sylow p-subgroup of V. Then $V = P \cdot \prod_{p_i \neq p} P_i$. Since V acts unipotently, its image under each φ_i such that $p_i \neq p$ is the identity; hence, $\overline{P}$ centralizes every $\overline{P_i}$ for which $p_i \neq p$. In view of corollary 5.1.2, the group P centralizes each P_i with $p_i \neq p$; i.e., it can be represented as a direct product of its Sylow subgroups and is nilpotent by lemma 5.1.5. We thus obtain a normal nilpotent subgroup of G which does not lie in N. This contradicts the definition of N. ⊠

Note that as a straightforward consequence we have $C_{\overline{G}}(\overline{N}) = \zeta(\overline{N}) = \overline{N}$. Moreover, $\overline{N} = F(\overline{G})$. Indeed, $\overline{N}$ is a normal abelian subgroup. Since $F(\overline{G})$ is nilpotent, by corollary 5.1.6 it acts unipotently on $\overline{N}$ and hence lies in $\overline{N}$. Therefore, we may assume that $G = \overline{G}$ and $F(G)$ is a product of elementary abelian groups. For this reason, to lighten notation we henceforth omit the overline. This consequence means in fact that the following holds:

Lemma 5.2.2. *If G is a finite solvable group then $F(G/\Phi(F(G))) = F(G)/\Phi(F(G))$.*

Lemma 5.2.3. *Let H be a nilpotent subgroup of G and let N_1 be a subgroup of N which is invariant under conjugation by H; moreover, the action of H on N_1 is not unipotent. Then the group $\langle H, N_1 \rangle$ is not nilpotent.*

PROOF. Indeed, suppose that the group $\langle H, N_1 \rangle$ is nilpotent. Then N_1 is its normal subgroup which is a direct product of elementary abelian groups. Therefore, the group $\langle H, N_1 \rangle$ acts on N_1 unipotently (by corollary 5.1.6), and so the group H acts on N_1 unipotently, which contradicts the hypothesis. ⊠

We continue the proof of the theorem. Let H_1 be the subset of all elements of H that act unipotently. Then H_1 is a normal subgroup of H. Indeed, closure with respect to inversion and conjugation is obvious; therefore, it suffices to check closure with respect to multiplication. Let $x, y \in H_1$ be arbitrary two elements. Then $|x^{\varphi_i}| = p_i^m$ and $|y^{\varphi_i}| = p_i^l$ for all $i = 1, \ldots, k$. Since H^{φ_i} is a nilpotent group, it can be represented as a direct product of its Sylow subgroups. In particular, the product of any two p_i-elements is again a p_i-element; i.e., $|(xy)^{\varphi_i}| = p_i^n$; hence, the element xy acts unipotently for all i and belongs therefore to H_1.

The subgroup $H \cap N$ is invariant under conjugation by H. Therefore, lemma 5.2.3 implies that H acts on $H \cap N$ unipotently. Since N is an elementary abelian group, we may consider the factor-group $N/(N \cap H) = Q_1 \times \cdots \times Q_k$, where $|Q_i| = p_i^{m_i}$ and $\exp(Q_i) = p_i$. By invariance of $N \cap H$ under conjugation by H, we may consider the induced homomorphisms $\phi_i : H \to GL(m_i, p_i) = GL(Q_i)$. For every i the group H^{ϕ_i} is nilpotent; therefore, it can be represented as $T_i \times U_i$, the direct product of its semisimple and unipotent parts. In view of lemma 5.1.8, $|T_i| < |Q_i|^\beta$. Demonstrate that $|H/H_1| \leqslant \prod_i |T_i|$ and, in consequence,

$$|H/H_1| \leqslant |N/(N \cap H)|^\beta. \tag{5.1}$$

Let x and y be two elements in H whose images in H/H_1 differ. Then there is an $i \in \{1, \ldots, k\}$ such that $x^{\phi_i} U_i \neq y^{\phi_i} U_i$. Indeed, otherwise the element xy^{-1} acts unipotently on $N/(N \cap H)$. Since this element acts unipotently also on $N \cap H$, it acts unipotently on the whole group N and belongs therefore to H_1. This implies that the images of x and y

in the group H/H_1 coincide, which contradicts the choice of these elements. To complete the proof of inequality (5.1), we need the following simple lemma.

Lemma 5.2.4. *Suppose that A is a finite set and $\psi_i : A \to A_i$ $(i = 1, \dots, n)$ are mappings such that, for arbitrary two distinct elements a and b in A, there is an i such that $a^{\psi_i} \neq b^{\psi_i}$. Then $|A| \leqslant |A_1| \cdot \ldots \cdot |A_n|$.*

Proof. By the hypothesis of the lemma, we can arrange an injective embedding of A into the Cartesian product $A_1 \times \cdots \times A_n$ by the following rule: $a \to (a^{\psi_1}, \dots, a^{\psi_n})$. It follows that $|A| \leqslant |A_1 \times \cdots \times A_n| = |A_1| \cdot \ldots \cdot |A_N|$, which completes the proof of the lemma. ⊠

To finish the proof of inequality (5.1), observe that there are mappings of the elements of the group H/H_1 into the cosets of the subgroups U_i in the groups H^{ϕ_i} which satisfy the hypothesis of lemma 5.2.4. Therefore, $|H/H_1| \leqslant \prod_i |H^{\phi_i} : U_i| = \prod_i |T_i|$, and inequality (5.1) is proven.

We now validate the inequality

$$|G/N : H_1N/N|^2 > |H_1N/N| = |H_1/(H \cap N)|. \tag{5.2}$$

To this end, we consider the group $C_i = C_G(\prod_{j \neq i} P_j)/N$. Since $C_G(N) = N$, we have $C_i \cap \langle C_j | j \neq i \rangle = \{e\}$. Furthermore, it is clear that each group C_i is normal in G/N, and we can hence consider the subgroup $C = C_1 \times \cdots \times C_k$ of the group G/N. Since the group H_1 acts unipotently (and is itself nilpotent), the factor-group $H_1N/N \cong H_1/(H \cap N)$ (obviously, $H \cap N = H_1 \cap N$) can be represented as a direct product of its Sylow p_i-subgroups: $H_1N/N = H_{p_1} \times \cdots \times H_{p_k}$. It follows from the proof of Lemma 5.2.1 that $H_{p_i} \leq C_i$.

Next, since $C_i \trianglelefteq G/N$, there are no nontrivial normal p_i-subgroups in C_i. Otherwise the largest of these subgroups is automorphism admissible and hence is a nontrivial normal subgroup of G/N acting unipotently. This contradicts lemma 5.2.1. Thus, corollary 5.1.10 implies that $|C_i : H_{p_i}|^2 > |H_{p_i}|$. Combining these inequalities for all i, we obtain

$$|G/N : H_1N/N|^2 \geqslant |C : H_1N/N|^2 > |H_1N/N| = |H_1/(H \cap N)|,$$

completing the proof of (5.2).

To finish the proof of the main theorem, we need two equalities that are easy consequences of Lagrange theorem:

$$|G : H| = |G : HN| \cdot |HN : H| \overset{1}{=} |G/N : HN/N| \cdot |N/(N \cap H)| \tag{5.3}$$

Here in step 1 we use the fact that every element of HN can be written as $n \cdot h$, with $n \in N$ and $h \in H$. Therefore, every coset of H can be written as nH for some $n \in N$, and coincidence of two cosets n_1H and n_2H means that $n_2^{-1}n_1 \in H \cap N$. In consequence, $|NH : H| = |N : (N \cap H)| = |N/(N \cap H)|$ (the group N is abelian). Next,

$$\begin{aligned} |G/N : H_1N/N| = |G/N : HN/N| \cdot |HN/H_1N| &\overset{2}{=} \\ = |G/N : HN/N| \cdot |H/H_1| &= |G/N : H_1N/N|. \end{aligned} \tag{5.4}$$

Here the proof of step 2 bases on the fact that $H \cap N = H_1 \cap N$ and, in consequence, $|HN/H_1N| = |HN/N : H_1N/N| = |H/(H \cap N)|/|H_1/(H_1 \cap N)| = |H|/|H_1| = |H/H_1|$.

Now, we derive the final estimate:

$$
\begin{aligned}
|G:N| = |G/N : HN/N| \cdot |HN/N : H_1N/N| \cdot |H_1N/N| = \\
= |G/N : HN/N| \cdot |H/H_1| \cdot |H_1N/N| \overset{3}{<} \\
< |G/N : HN/N| \cdot |N/(N \cap H)|^{\beta} \cdot |G/N : H_1N/N|^2 \overset{4}{=} \\
= |G/N : HN/N| \cdot |N/(N \cap H)|^{\beta} \cdot |G/N : HN/N|^2 \cdot |H/H_1|^2 \overset{5}{<} \\
< |G/N : HN/N|^3 \cdot |N/(N \cap H)|^{\beta} \cdot |N/(N \cap H)|^{2\beta} \overset{6}{\leqslant} \\
\leqslant |G/N : HN/N|^3 \cdot |N/(N \cap H)|^{3\beta} < |G:H|^5.
\end{aligned}
$$

Here step 3 is obtained by applying (5.1) and (5.2) to the second and third factors respectively. Step 4 results from applying (5.4) to the last factor. Step 5 follows again from (5.1). Finally, step 6 ensues from (5.3) and the inequality $|G/N : HN/N| \geqslant 1$ which follows from the fact that $3 < 3\beta < 5$. The proof of theorem 1.1.4 is over.

Chapter 6

Some corollaries

§1 Abelian ABA- factorization of finite simple groups

We prove theorem 1.1.2 in this section.

Proof. Clearly, if abelian subgroups of G satisfy $|A|^3 < |G|$, then G fails to be represented as ABA. We thus need to consider only groups $A_1(q)$. In so doing, we distinguish between the cases with q even and q odd.

Let q be odd, $q \geqslant 7$. Then the orders of maximal abelian subgroups are equal to q, $\frac{q+1}{2}$, $\frac{q-1}{2}$. If $A_1(q) = ABA$, for the orders of A and B we face the following two options: $|A| = |B| = q$ or $|A| = q$, $|B| = \frac{q+1}{2}$. By routine computations, using the canonical form of elements in $A_1(q)$, we conclude that if u is a nonidentity unipotent element then $C_{A_1(q)}(u) = U$, where U is a unipotent subgroup of $A_1(q)$. For every Sylow p-subgroup P and for any element x in $A_1(q)$, therefore, $P \cap P^x$ is equal either to 1 or to P.

We proceed to consider both options for A and B. Seek an order of the set AbA, where b is some element in B. We have $|AbA| = |AbAb^{-1}| = \frac{|A|^2}{|A\cap A^{b^{-1}}|} = \begin{cases} |A| \\ |A|^2 \end{cases}$.

If $A_1(q) = ABA$, we arrive at the following two systems of equations (for the first and second versions, respectively):

$$\begin{cases} qx + y = \frac{q^2-1}{2}, \\ x + y = q \end{cases}$$

and

$$\begin{cases} qx + y = \frac{q^2-1}{2}, \\ x + y = \frac{q+1}{2}, \end{cases}$$

where x and y are integers. It is not hard to see that neither system has an integer-valued solution, for any q. Therefore, $L_2(q)$ is not represented as ABA if q is odd.

Let q be even. Then, for the orders of A and B, there are four options for which $|G| \leqslant |A| \cdot |B| \cdot |A|$. These are the following pairs: $(q, q+1)$, (q, q), $(q+1, q-1)$, and $(q+1, q)$. The first two are treated the same way as for q odd, and we have $G \neq ABA$ for them. For the other two, we make use of the fact that $A_1(2^t) \cong SL_2(2^t)$. If the order of A is $q+1$, then A is conjugate in $SL_2(2^{2t})$ to a subgroup of matrices of the form

$$\begin{pmatrix} \alpha\, 0 \\ 0\, \alpha^q \end{pmatrix}.$$

Thus $C_{SL_2(q)}(a) = A$ for a nonidentity element a in A; hence, $A \cap A^x$ is equal either to 1 or to A. Also, $|N_{SL_2(q)}(A)| = 2(q+1)$. Using the same argument as for q odd, we arrive at the following two systems (for the third and fourth versions, respectively):

$$\begin{cases} (q+1)x + y = q(q-1), \\ x + y = q+1 \end{cases}$$

and

$$\begin{cases} (q+1)x + y = q(q-1), \\ x + y = q+1. \end{cases}$$

The system has no integer-valued solution in the third case, but has in the fourth — $x = q-2$, $y = 2$. We have $|N_{SL_2(q)}(A)| = 2(q+1)$, and so $SL_2(q) = ABA$, which proves theorem 1.1.2.

⊠

§2 Large normal nilpotent subgroups of finite groups

As a corollary to theorem 1.1.4, we obtain a general answer to the question we raised in the beginning of the paper.

Theorem 6.2.1. *Let G be a finite group. If G has a nilpotent subgroup of index n then it has a normal nilpotent subgroup of index at most n^c for some absolute constant c.*

Proof. By [2, theorem 2.13], the group G has a normal solvable subgroup R of index at most n^{c_1} for some absolute constant c_1. Let H be a nilpotent subgroup of index n appearing in the hypothesis of the theorem. Then $R \cap H$ is a nilpotent subgroup of index at most n in R. By theorem 1.1.4, $|R : F(R)| < n^5$. Since the Fitting subgroup is characteristic, it is normal in G and $|G : F(R)| < n^{c_1+5}$. ⊠

Remark. We proved theorem 1.1.4 without appealing to the classification of finite simple groups. The proof of theorem 2.13 in [2] leans essentially on the theorem of classification of finite simple groups. Using the proof of Theorem 2.13 in [2], we can obtain an estimate for the constant c_1 (in the proof of Theorem 6.2.1):

$$c_1 \leqslant \frac{\beta+1}{1-\alpha} + \frac{2}{(1-\alpha)\log_2 60}.$$

Here the constants α and β are defined as follows:

$\alpha < 1$ is an absolute constant such that the inequality $|N| \leqslant |G|^\alpha$ holds for every finite nonabelian simple group G and every nilpotent subgroup N of G;

β is an absolute constant such that the inequality $|\operatorname{Out}(G)| \leqslant |G|^\beta$ holds for every finite nonabelian simple group G.

By theorem 1.1.3 we may take $\frac{1}{2}$ as α; β can be taken to be $\frac{1}{2}$ as well. Thus, the constant c in theorem 6.2.1 is at most 9.

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