New rewriting system for the braid group $B_4$

LEONID BOKUT$^1$ AND ANDREI VESNIN$^{1,2}$

$^1$Sobolev Institute of Mathematics, Novosibirsk 630090, Russia
$^2$School of Mathematical Sciences, Seoul National University, Seoul 151-747, Korea

To 60-th birthday of a pioneer of Gröbner bases Prof. B. Buchberger

Abstract
Using presentations of the braid groups $B_3$ and $B_4$ as towers of HNN extensions of the free group of rank 2, we obtain normal forms, Gröbner bases and rewriting systems for these groups.

KEYWORDS: braid group, normal form, Gröbner basis

1. Introduction
In this note we obtain Gröbner bases as well as rewriting systems for braid groups $B_3$ and $B_4$. Braid groups are subject of the intensive studying in group theory and low-dimensional topology. We refer to (Birman 1974) for the basic properties of braid groups. It was shown by P. Dehornoy that braid groups are left-orderable (Dehornoy 2000), and different kinds of normal forms and rewriting systems for braid groups can be found in (Birman et al. 1998, Elrifai Morton 1994, Garside 1969, Garber et al. 2002, Hermiller Meier 1999, Markov 1945, Thurston 1992).

Our interest in Gröbner bases for braid groups is motivated by the close relation between non-commutative Gröbner bases and rewriting systems for semigroups. This relation is a kind of a folklore, and was fully described, for example, in (Heyworth 2000). As an introduction to string rewriting we refer to (Book Otto 1993). The basis facts about Gröbner bases will be presented in Section 2.

One of aims of this note is to develop and to demonstrate a method of constructing of Gröbner bases, normal forms, and rewriting systems for groups, presented as towers of HNN extensions of free groups. We realize this method for braid groups $B_3$ and $B_4$.

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In Section 3, applying the Magnus-Moldovanski rewriting procedure (see Lyndon Schupp (1977)) for $B_3$ and $B_4$ (which is 3-relator group rather than to 1-relator classical case) we obtain presentations of these groups as towers of HNN extensions of the free group of rank 2. Obtained presentations are closely related to the presentations of commutators $B'_3$ and $B'_4$ which have been find by E. Gorin and V. Lin (see Gorin Lin (1969)) using the Reidemeister-Shreier method. So, we will refer to the generators as Gorin-Lin generators.

These presentations lead to the standard normal forms in $B_3$ and $B_4$, and to the standard rewriting systems for them in the sense of (Bokut 1966, 1967, 1980). We show it in Section 4, using the Gröbner–Shirshov bases technic unlike of a group-theoretic technic of (Bokut 1966, 1967). More precisely, we give the Gröbner–Shirshov basis of $B_4$ (and as a corollary, for $B_3$) in Gorin-Lin generators and relative to an appropriate order of group words, the tower order. This order was implicitly used in (Bokut 1966, 1967). The applications of similar technique to Coxeter groups and Novikov and Boon groups can be found in (Bokut Shiao 2001, 2002).

From the normal form, it readily follows some known properties of $B_4$ proved in the above mentioned paper by E. Gorin and V. Lin, in particular, the presentation of the commutator group $B'_4$ and its description as the semidirect product of two free groups of rank two. Moreover, the relation of obtained presentations of braid groups with cyclically presented groups, such as Sieradski groups and Fibonacci groups is discussed.

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2. Non-Commutative Gröbner bases

In this section we recall some facts about non-commutative Gröbner bases which are also known in the literature as Gröbner–Shirshov bases (see, for example, Ufnarovski (1998)).

Let $X$ be a linearly ordered set, $k$ a field, $k(X)$ the free associative algebra over $X$ and $k$. On the set $X^*$ of words we impose a well order “$>$” that is compatible with the concatenations of words. For example, it may be the deglex order (compare two words first by degrees and then lexicographically) or the tower order below. Any polynomial $f \in k(X)$ has the leading word $\bar{f}$ relative to “$>$”.

We say that $f$ is monic if $\bar{f}$ occurs in $f$ with coefficient 1. By a composition of intersection $(f, g)_w$ of two monic polynomials relative to some word $w$, such that $w = \bar{f}b = a\bar{g}$, $\deg(\bar{f}) + \deg(\bar{g}) > \deg(w)$, one means the following polynomial

$$(f, g)_w = fb - ag.$$

By composition of including $(f, g)_w$ of two monic polynomials, where $w = \bar{f} = a\bar{g}b$, one means the following polynomial

$$(f, g)_w = f - a\bar{g}b.$$
In the last case the transformation
\[ f \rightarrow (f, g)_w = f - agb \]
is called the \textit{elimination of the leading word (ELW)} of g in f.

A composition \((f, g)_w\) is called \textit{trivial} relative to some \(R \subset k\langle X\rangle\) and \(w\) (we write it as \((f, g)_w \equiv 0 \mod (R, w)\)) if
\[ (f, g)_w = \sum \alpha_i a_i t_i b_i, \]
where \(\alpha_i \in k\), \(t_i \in R\), \(a_i, b_i \in X^*\), and \(a_i t_i b_i < w\). In particular, if \((f, g)_w\) goes to zero by the ELW’s of \(R\) then \((f, g)_w\) is trivial relative to \(R\).

For two polynomials \(f_1\) and \(f_2\) we write
\[ f_1 \equiv f_2 \mod (R, w) \]
if and only if
\[ f_1 - f_2 \equiv 0 \mod (R, w). \]

A subset \(R\) of \(k\langle X\rangle\) is called \textit{Gröbner–Shirshov basis} if any composition of polynomials from \(R\) is trivial relative to \(R\).

By \(\langle X | R \rangle\), the algebra with generators \(X\) and defining relations \(R\), we will mean the factor-algebra of \(k\langle X\rangle\) by the ideal generated by \(R\).

The following lemma goes back to the Poincare-Birkhoff-Witt theorem, the Diamond Lemma of M.H.A. Newman (Newman 1942), the Composition Lemma of A.I. Shirshov (Shirshov 1962) (see also Bokut (1972, 1976), where this Composition Lemma was formulated explicitly and in a current form), the Buchberger’s Theorem (Buchberger (1965), published in Buchberger (1970)), the Diamond Lemma of G. Bergman (Bergman 1978) (this lemma was also known to P.M. Cohn (see, for example, Cohn (1966)) and some historical comments to Chapter “Gröbner basis” in (Eisenbud 1995)):

**Composition–Diamond Lemma.** \(R\) is a Gröbner–Shirshov \textit{basis} if and only if the set
\[ PBW(R) = \{ u \in X^* \mid u \neq afb, \text{ for any } f \in R \} \]
of \(R\)-reduced words consists of a linear basis of the algebra \(\langle X | R \rangle\).

The set \(PBW(R)\) will be called the \textit{PBW-basis} of \(\langle X | R \rangle\) relative to a Gröbner–Shirshov \textit{basis} \(R\).

If a subset \(R\) of \(k\langle X\rangle\) is not a Gröbner–Shirshov basis then one can add to \(R\) all non trivial compositions of polynomials of \(R\), and continue this process (infinitely) many times in order to have a Gröbner–Shirshov basis \(R^{\text{comp}}\) that contains \(R\).

A Gröbner–Shirshov \textit{basis} \(R\) is called \textit{reduced} if any \(s \in R\) is a linear combination of \(R \setminus \{s\}\)–reduced words. Any ideal of \(k\langle X\rangle\) has a unique reduced Gröbner–Shirshov \textit{basis}.
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If $R$ is a set of “semigroup relations” (that is, polynomials of the form $u - v$, where $u, v \in X^*$), then any non trivial composition will have the same form. As the result the set $R^{\text{comp}}$ consists of semigroup relations too.

Let $A = \text{smg}(X|R)$ be a semigroup presentation. Then $R$ is a subset of $k\langle X \rangle$ and one can find a Gröbner–Shirshov basis $R^{\text{comp}}$. The last set does not depend of $k$, and consists of semigroup relations. We will call $R^{\text{comp}}$ to be a Gröbner–Shirshov basis of $A$. It is the same as a Gröbner–Shirshov basis of the semigroup algebra $kA = \langle X|R \rangle$.

The same terminology is valid for any group presentation meaning that we include in this presentation all trivial group relations of the form $xx^{-1} = 1, x^{-1}x = 1, x \in X$.

3. Presentations of groups $B_3$ and $B_4$

In this section we will obtain presentations of braid groups as towers of HNN extensions of the free group of rank two.

We start from the consideration of the 3-strand braid group $B_3$ with the following presentation:

$$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle.$$ (1)

Let us denote $t = \sigma_1$ and consider $y_2$ such that $\sigma_2 = y_2t$, i.e. $y_2 = \sigma_2\sigma_1^{-1}$. Then we have

$$B_3 \cong \langle y_2, t \mid y_2ty_2 = ty_2t \rangle.$$

Introducing notation $y_{2(i)} = t^iy_2t^{-i}$ for $i = 1, 2$, we get

$$B_3 \cong \langle y_{2(1)}, y_{2(2)}, y_{2(1)}y_{2(2)} = y_{2(1)}, \ y_{2(1)}t = ty_{2(2)}, \ y_{2(2)}t = ty_{2(1)} \rangle.$$

Eliminating $y_{2(2)}$ using the first relation, we will obtain:

$$B_3 \cong \langle y_{2(1)}, y_{2(1)}t = ty_{2(2)}, \ y_{2(1)}^{-1}y_{2(1)}t = ty_{2(1)} \rangle$$

and finally,

$$B_3 \cong \langle y_{2(1)}, t \mid y_{2(1)}t = ty_{2(2)}, \ y_{2(1)}^{-1}t = ty_{2(1)} \rangle,$$

where we used the first relation to modify the second.

Let us rewrite this relation using notations $y_2 = t_2$ and $y_{2(1)} = t_1$:

$$B_3 \cong \langle t_1, t_2 \mid t_1t = tt_2, \ t_2t = tt_1^{-1} \rangle.$$ (2)

Thus, we have the following extension:

$$\langle t_1, t_2 \rangle \subset B_3 = \langle t_1, t_2, t \rangle,$$ (3)

that can be regarded as an HNN-extension (by the conjugation automorphism $t$) of the free group with two generators. The relation of these generators with the standard generators of $B_3$ is the following: $t_2 = \sigma_2\sigma_1^{-1}$ and $t_1 = \sigma_1\sigma_2\sigma_1^{-2}$.

Braids corresponding to $t$, $t_1$, and $t_2$ are presented in the following figure.
As the result, the kernel of the homomorphism from \( B_3 \) to \( \langle t \rangle \) defined by \( t \mapsto t, t_1 \mapsto 1, t_2 \mapsto 1 \), is the free group \( \langle t_1, t_2 \rangle \). This kernel coincides with the commutant \( B_3' \), that was also shown in (Gorin Lin 1969) by using of the Reidemeister–Schreier method. So, we will refer to \( t, t_1 \) and \( t_2 \) as Gorin-Lin generators of \( B_3 \).

From the defining relations (2) and their inverses we obtain relations:

\[
\begin{align*}
t_2 t &= t t_2 t_1^{-1}, & t_1 t^{-1} &= t^{-1} t_2^{-1} t_1, \\
t_1 t &= t t_2, & t_2 t^{-1} &= t^{-1} t_1,
\end{align*}
\]

where we used

\[
t_2 t = t t_2 t_1^{-1} \iff t_2 t = t_1 t t_1^{-1} \iff t_1^{-1} t_2 t = t t_1^{-1}, \iff t_1 t^{-1} = t^{-1} t_2^{-1} t_1.
\]

In the next section we will prove that the following transformations give rise the rewriting system for \( B_3 \) (that will follows from the rewriting system for \( B_4 \) and embedding \( B_3 \subset B_4 \)):

\[
\begin{align*}
(t_2)^{\pm 1} t &\rightarrow t (t t_1^{-1})^{\pm 1}, & (t_1)^{\pm 1} t^{-1} &\rightarrow t^{-1} (t_2^{-1} t_1)^{\pm 1}, \\
(t_1)^{\pm 1} t &\rightarrow t (t_2)^{\pm 1}, & (t_2)^{\pm 1} t^{-1} &\rightarrow t^{-1} (t_1)^{\pm 1}.
\end{align*}
\]

It gives the following normal form for \( B_3 \):

\[
t^n V(t_1, t_2)
\]

where \( n \in \mathbb{Z} \) and \( V(t_1, t_2) \) denotes a group word in \( t_1, t_2 \).

We would like to point out the following relation of the braid group presentation (2) with cyclically presented groups (in sense of Johnson (1980)).

Consider the conjugation action of \( t \) on the group \( B_3' = \langle t_1, t_2 \rangle \). Let us denote \( a_0 = t_2 \) and \( a_i = t a_0 t^{-i} \) for \( i \in \mathbb{Z} \). In particular, we get \( a_1 = t_1 \) and \( a_{-1} = t_2 t_1^{-1} = a_0 a_1^{-1} \). Therefore, for each \( i \) we have \( a_i a_{i+2} = a_{i+1} \), and the following group presentation with infinite number of generators naturally arises:

\[
G_\infty = \langle a_i, i \in \mathbb{Z} \mid a_i a_{i+2} = a_{i+1}, \ i \in \mathbb{Z} \rangle
\]

For a reader convenience we recall the expression of \( a_i \) in terms of the Artin generators (1):

\[
a_i = \sigma_1^i \sigma_2 \sigma_1^{-i+1}.
\]

Remark that the “truncated” version of this group, i.e.

\[
G_n = \langle a_1, \ldots, a_n \mid a_i a_{i+2} = a_{i+1}, \ i = 1, \ldots, n \rangle,
\]
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where all suffices are taken by mod $n$, is known as the Sieradski group (see, for example, Cavicchioli et al. (1998)) and is the fundamental group of the $n$-fold cyclic branched covering of the 3-sphere, branched over the trefoil knot.

Now let us consider the 4-strand braid group $B_4$ with the following presentation:

$$B_4 = \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_2 \sigma_1 \sigma_2 = \sigma_1 \sigma_2 \sigma_1, \quad \sigma_3 \sigma_2 \sigma_3 = \sigma_2 \sigma_3 \sigma_2, \quad \sigma_3 \sigma_1 = \sigma_1 \sigma_3 \rangle. \quad (4)$$

Let us denote $t = \sigma_1$. Consider $y_3$ such that $\sigma_3 = y_3 t$, i.e. $y_3 = \sigma_3 \sigma_1^{-1}$, and $y_2$ such that $\sigma_2 = y_2 t$, i.e. $y_2 = \sigma_2 \sigma_1^{-1}$. Then we have

$$B_4 \cong \langle y_2, y_3, t \mid y_2 t y_2 = t y_2, \quad y_3 t y_3 = y_2 t y_3 t y_2, \quad y_3 t = t y_3 \rangle.$$

As well as above, let us introduce $y_{2(i)} = \iota^i y_{2-i}$ for $j = 2, 3$ and $i \in \mathbb{Z}$. Then

$$B_4 \cong \langle y_2, y_{2(1)}, y_{2(2)}, y_3, t \mid y_{3 y_{2(1)} y_3} = y_2 y_{3 y_{2(2)}}, \quad y_3 t = t y_3, \quad y_{2(1)} t = t y_2, \quad y_{2(2)} t = t y_{2(1)} \rangle,$$

where we used that $t$ and $y_3$ commute. Eliminating $y_{2(2)}$ using the first defining relation, we will obtain:

$$B_4 \cong \langle y_2, y_{2(1)}, y_3, t \mid y_3 y_{2(1)} y_3 = y_2 y_{3 y_2^{-1}} y_{2(1)}, \quad y_3 t = t y_3, \quad y_{2(1)} t = t y_2, \quad y_{2(1)}^{-1} t y_{2(1)} = t y_{2(1)} \rangle.$$

Denoting $t_1 = y_{2(1)}, t_2 = y_2$ and $b = y_3$ we get

$$B_4 \cong \langle t_1, t_2, b \mid b t_1 b = t_2 b t_2^{-1} t_1, \quad b t = t b, \quad t_1 t = t t_2, \quad t_2^{-1} t_1 t = t t_1 \rangle,$$

and so,

$$B_4 \cong \langle t_1, t_2, b \mid t_1 b = b^{-1} t_2 b t_2^{-1} t_1, \quad b t = t b, \quad t_1 t = t t_2, \quad t_2 t = t t_2 t_1^{-1} \rangle,$$

where we used the third relation to modify the forth relation. Denoting $a = t_1 b t_1^{-1}$, we get

$$B_4 \cong \langle a, b, t_1, t_2, t \mid t_1 t = t t_2, \quad t_2 t = t t_2 t_1^{-1}, \quad b t = t b, \quad a t_1 = t_1 b, \quad a = b^{-1} t_2 b t_2^{-1} \rangle. \quad (5)$$

Generators $t, t_1, t_2, a, b$ will be referred to as Gorin-Lin generators of $B_4$.

Using above defining relations we have

$$a t = t_1 b t_1^{-1} t = t_1 b t t_2^{-1} = t_1 t b t_2^{-1} = t t_2 b t_2^{-1} = t b a. \quad (6)$$

Let us multiply both sides of the relation

$$t_1 b = b^{-1} t_2 b t_2^{-1} t_1$$
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from the left by \( t \), and remark that after that the left part can be modified as
\[
   t t_1 b = t_2^{-1} t_2 t b = t_2^{-1} t_1 t b = t_2^{-1} t_1 b t = t_2^{-1} a t_1 t,
\]
and the right part can be modified as
\[
   t b^{-1} t_2 b t_2^{-1} t_1 = b^{-1} t_2 b t_2^{-1} t_1 = b^{-1} t_1 t b t_2^{-1} t_1 = b^{-1} t_1 b t^{-1} t_1
\]

Thus,
\[
   t_2^{-1} a = b^{-1} a t_2^{-1},
\]
that gives us
\[
   a t_2 = t_2 b^{-1} a. \tag{7}
\]

Using this result we have
\[
   a = b^{-1} t_2 b t_2^{-1} \iff b a t_2 = t_2 b \iff b t_2 b^{-1} a = t_2 b \iff b t_2 = t_2 b a^{-1} b. \tag{8}
\]

Now let us conjugate both sides of the obtained relation by \( t \). Then from the left part of the relation we will get
\[
   t b t_2 t^{-1} = b t t_2 t^{-1} = b t_1,
\]
and from the right part of the relation we will get
\[
   t t_3 b a^{-1} b t^{-1} = t_1 t b a^{-1} t^{-1} b = t_1 b t^{-1} b^{-1} t b t^{-1} b = t_1 b a^{-1} b^2.
\]
Hence
\[
   b t_1 = t_1 b a^{-1} b^2. \tag{9}
\]

Summarizing (5), (6), (7), (8) and (9) we get

**Lemma 3.1:** The following relations are valid for Gorin-Lin generators of \( B_4 \):
\[
   \begin{align*}
   t_2 t &= t t_2 t_1^{-1} \quad t_1 t &= t t_2, \quad b t &= t b, \quad a t &= t b a, \\
   a t_1 &= t_1 b, \quad b t_1 &= t_1 b a^{-1} b^2, \quad a t_2 &= t_2 b^{-1} a, \quad b t_2 &= t_2 b a^{-1} b,
   \end{align*}
\]
and for their inverses:
\[
   \begin{align*}
   t_1 t^{-1} &= t^{-1} t_2 t_1, \quad t_2 t^{-1} &= t^{-1} t_1, \quad b t^{-1} &= t^{-1} b, \quad a t^{-1} &= t^{-1} b^{-1} a, \\
   b t_1^{-1} &= t_1^{-1} a, \quad a t_1^{-1} &= t_1^{-1} a b^{-1} a, \quad b t_2^{-1} &= t_2^{-1} b a, \quad a t_2^{-1} &= t_2^{-1} b a^2.
   \end{align*}
\]

Therefore, we have the description of \( B_4 \) as a tower of HNN extensions of the free group of rank two:
\[
   \langle a, b \rangle \subset \langle a, b, t_1, t_2 \rangle \subset B_4 = \langle a, b, t_1, t_2, t \rangle.
\]

The correspondence between Gorin-Lin generators and the standard generators of \( B_4 \) is the following:
\[
   t = \sigma_1, \quad t_1 = \sigma_1 \sigma_2 \sigma_1^{-2}, \quad t_2 = \sigma_2 \sigma_1^{-1}, \quad a = \sigma_1 \sigma_2 \sigma_1^{-1} \sigma_3 \sigma_2^{-1} \sigma_1^{-1}, \quad b = \sigma_3 \sigma_1^{-1}.
\]

Remark that \( t, t_1 \) and \( t_2 \) generate such a subgroup of \( B_4 \) with the presentation (2) that gives us \( B_3 \). Braids corresponding to \( t, t_1, t_2, a \) and \( b \) are presented in the following figure.
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\[ t : \quad t_1 : \quad t_2 : \quad a : \quad b : \]

In the next section we will find a rewriting system for $B_4$. As the result, a normal form for $B_4$ is given by expression

\[ t^n V(t_1, t_2) W(a, b), \]

where $n \in \mathbb{Z}$, and $V(t_1, t_2)$, $W(a, b)$ are free group words in alphabets $\{t_1, t_2\}$ and $\{a, b\}$, respectively. The commutant $B'_4$, the kernel of the homomorphism from $B_4$ to $\langle t \rangle$, defined by

\[ t \mapsto t, \quad t_1 \mapsto 1, \quad t_2 \mapsto 1, \quad a \mapsto 1, \quad b \mapsto 1 \]

is the semi-direct product of free groups $\langle t_1, t_2 \rangle$ and $\langle a, b \rangle$. Here $\langle t_1, t_2 \rangle$ is the commutant of $B_3$. The kernel of the homomorphism of $B_4$ to $B_3$, defined by

\[ \sigma_1 \mapsto \sigma_1, \quad \sigma_2 \mapsto \sigma_2, \quad \sigma_3 \mapsto \sigma_1, \]

that is by

\[ t \mapsto t, \quad t_1 \mapsto t_1, \quad t_2 \mapsto t_2, \quad a \mapsto 1, \quad b \mapsto 1, \]

is the free group $\langle a, b \rangle$. Remark that in (Gorin Lin 1969) all these facts were proved using the Reidemeister-Shreier method, where $t^{-1}at$ and $b$ were chosen as the generators of this free group.

Now let us point out the connection of relations from Lemma 3.1 with Fibonacci groups $F(2, n)$ studied by many authors.

Consider the conjugation action of $t_1$ on $\langle a, b \rangle$:

\[ t_1^{-1}at_1 = b, \quad t_1^{-1}bt_1 = ba^{-1}b^2. \]

Let us denote $z_0 = b$ and $z_i = t_1^{-1}z_0 t_1^i$ for $i \in \mathbb{Z}$ . Thus, we get $a = z_{-1}$ and $z_1 = z_0 z_{-1}^{-1} z_0^2$. Therefore for $i \in \mathbb{Z}$ we have

\[ z_{i+1} z_i^{-1} z_{i+1}^2 = z_{i+2}, \]

and the following group with infinite number of generators naturally arise:

\[ H_\infty = \langle z_i, i \in \mathbb{Z} \mid z_{i+1} z_i^{-1} z_{i+1}^2 = z_{i+2}, \quad i \in \mathbb{Z} \rangle. \]

Remark that this group presentation coincides with the presentation of the commutator subgroup of the figure-eight knot group obtained in (Burde Zieschang 1985, p. 35) and action of $t_1$ on $a$ and $b$ corresponds to the presentation of the figure-eight knot as a fibred knot (Burde Zieschang 1985, p. 73).
Similar to (Kim et al. 2000), rewrite the defining relations as
\[(z_i^{-1}z_{i+1})z_{i+1} = (z_{i+1}^{-1}z_{i+2}), \quad i \in \mathbb{Z}\]
and denote \(x_{2i-1} = z_i\) and \(x_{2i} = z_i^{-1}z_{i+1}\) for \(i \in \mathbb{Z}\). Then the above relations can be rewritten as \(x_{2i} = x_{2i-1}x_{2i+1}\) and \(x_{2i}x_{2i+1} = x_{2i+2}\). Then
\[H_{\infty} \cong \langle x_{2i-1}, x_{2i} \mid x_{i}x_{i+1} = x_{i+2}, \quad i \in \mathbb{Z} \rangle.\]
For a reader convenience we recall the expression of \(x_i\) in terms of the Gorin-Lin generators:
\[x_{2i-1} = t_i^{-1}bt_1, \quad x_{2i} = t_i^{-1}b^{-1}t_1^{-1}bt_1t_1, \quad i \in \mathbb{Z}.\]
Thus, relations \(x_{2i-1}x_{2i} = x_{2i+1}\) follow immediately, and relations \(x_{2i}x_{2i+1} = x_{2i+2}\) follow from the relation \(bt_1 = t_1bt_1b^{-1}t_1^{-1}b^2\).

The “truncated” version of this group is well-known as the Fibonacci group:
\[F(2, 2n) = \langle x_1, \ldots, x_{2n} \mid x_{i}x_{i+1} = x_{i+2}, \quad i = 1, \ldots, 2n \rangle\]
where all indices are taken by mod 2n. For any \(n \geq 2\) the group \(F(2, 2n)\) is the fundamental group of a 3-dimensional manifold (see Helling et al. (1998)) which can be described as the \(n\)-fold cyclic branched cover of the 3-sphere branched over the figure-eight knot. From the above considerations one can easily obtain expressions of generators of Fibonacci groups in terms of Artin generators of \(B_4\).

4. Rewriting systems for \(B_3\) and \(B_4\)

With each of the relations of \(B_4\) from Lemma 3.1 we associate another relation in the natural way shown below. Thus, we will get pairs of relations as follows:
\[
\begin{align*}
 t_2t &= tt_2t_1^{-1} & & t_2^{-1}t &= tt_1t_2^{-1} \\
 t_1t &= tt_2 & & t_1^{-1}t &= tt_2^{-1} \\
 bt &= tb & & b^{-1}t &= tb^{-1} \\
 at &= tba & & a^{-1}t &= ta^{-1}b^{-1} \\
 at_1 &= t_1b & & a^{-1}t_1 &= t_1b^{-1} \\
 bt_1 &= t_1ba^{-1}b^2 & & b^{-1}t_1 &= t_1b^{-2}ab^{-1} \\
 at_2 &= t_2b^{-1}a & & a^{-1}t_2 &= t_2a^{-1}b \\
 bt_2 &= t_2ba^{-1}b & & b^{-1}t_2 &= t_2b^{-1}ab^{-1} \\
 t_1t_1^{-1} &= t^{-1}t_2^{-1}t_1 & & t_1^{-1}t &= t^{-1}t_1t_2 \\
 t_2t_1^{-1} &= t^{-1}t_1 & & t_2^{-1}t_1 &= t^{-1}t_1^{-1} \\
 bt_1^{-1} &= t^{-1}b & & b^{-1}t_1^{-1} &= t^{-1}b^{-1} \\
 at_1^{-1} &= t^{-1}b^{-1}a & & a^{-1}t_1^{-1} &= t^{-1}a^{-1}b \\
 bt_1^{-1} &= t_1^{-1}a & & b^{-1}t_1^{-1} &= t_1^{-1}a^{-1} \\
 at_2^{-1} &= t_2^{-1}a^2b^{-1}a & & a^{-1}t_2^{-1} &= t_2^{-1}a^{-1}b^{-1} \\
 bt_2^{-1} &= t_2^{-1}ba & & b^{-1}t_2^{-1} &= t_2^{-1}a^{-1}b^{-1} \\
 at_2^{-1} &= t_2^{-1}ba^2 & & a^{-1}t_2^{-1} &= t_2^{-1}a^{-2}b^{-1}
\end{align*}
\]
New rewriting system for the braid group $B_4$

Let $S$ be the set of above listed relations (10) – (25) together with trivial relations $xx^{-1} = 1$, and $x^{-1}x = 1$ for $x \in \{a, b, t_1, t_2, t\}$.

We will prove that $S$ is the Gröbner–Shirshov basis of $B_4$ relative to the following tower order of words. Let us order group words in $a, b$ by the deglex order. Any group word in $a, b, t_2$ has a form

$$u = u_0 t_2^{\varepsilon_1} \cdots u_k t_2^{\varepsilon_k} u_{k+1},$$

where $u_i \in \langle a, b \rangle$, $k \geq 0$, $\varepsilon_i = \pm 1$. Define

$$wt(u) = (k, u_0, t_2^{\varepsilon_1}, \ldots, u_k, t_2^{\varepsilon_k}, u_{k+1}).$$

Let us order $wt$’s lexicographically assuming $t^{-1} < t$. Let us define the tower order

$$u >_{tow} v \text{ if and only if } wt(u) >_{lex} wt(v).$$

In the same way we can define the tower order for group words with the extra letter $t_1$ and then for group words with the extra letter $t$.

**Theorem 4.1:** The above described set $S$ is the reduced Gröbner–Shirshov basis for $B_4$ in the Gorin-Lin generators relative to the tower order of group words.

**Proof:** To prove the statement we need to check that all compositions (in the sense of Section 2) of pairs of elements of $S$ are trivial. Here we will do it for few of them. For all others similar considerations are used, but we omit them here.

Let us consider the composition $(10_r) \land (16_r)$ of an intersection of left relations in (10) and (16) relative to a word $w = at_2 t$. We have $g = t_2 t - tt_2 t_1^{-1}$ with the leading word $\bar{g} = t_2 t$ and $f = at_2 - t_2 b^{-1} a$ with the leading word $f = at_2$. Therefore $w = a \bar{g} = ft$. Then

$$(10_r) \land (16_r) = (f, g)_w = ft - ag = (at_2 - t_2 b^{-1} a)t - a(t_2 t - tt_2 t_1^{-1})$$

$$= att_2 t_1^{-1} - t_2 b^{-1} at \equiv tbat_2 t_1^{-1} - t_2 b^{-1} tba$$

$$\equiv tbt_2 b^{-1} a t_1^{-1} - t_2 b^{-1} tba \equiv tt_2 b a^{-1} b^{-1} a t_1^{-1} - t_2 t a$$

$$\equiv tt_2 b t_1^{-1} - tt_2 t_1^{-1} a \equiv tt_2 t_1^{-1} a - tt_2 t_1^{-1} a \equiv 0,$$

where we used relations from the set $S$ for the ELW’s of $S$.

Now, let us check that the composition $(10_r) \land (24_r)$ of right relation of (10) and left relation of (24) is trivial with respect to $S$ and $w = b t_2^{-1} t$. We have $g = t_2^{-1} t - tt_2 t_1^{-1}$ with the leading word $\bar{g} = t_2^{-1} t$ and $f = b t_2^{-1} - t_2^{-1} b a$ with the leading word $f = b t_2^{-1}$. Therefore $w = b \bar{g} = ft$. Then

$$(10_r) \land (24_r) = (f, g)_w = ft - bg = (bt_2^{-1} - t_2^{-1} ba)t - b(t_2^{-1} t - tt_2 t_1^{-1})$$

$$= btt_2^{-1} - t_2^{-1} bat \equiv tbt_2 t_1^{-1} - t_2^{-1} tba$$

$$\equiv tt_1 b a^{-1} b^{-1} t_2^{-1} - t_2^{-1} t^2 b a \equiv tt_1 b a^{-1} b^{-1} t_2^{-1} b a - tt_2 t_1^{-1} t_2^{-1} b a$$

$$\equiv tt_1 b a^{-1} b b a - tt_2 t_1^{-1} b a \equiv tt_1 b t_2^{-1} a^{-2} b^{-1} b a - tt_1 t_2^{-1} b^2 a$$

$$\equiv tt_1 b a^{-1} b a \equiv tt_1 b t_2^{-1} b^2 a - tt_1 t_2^{-1} b^2 a \equiv 0,$$
where we used relations from the set $S$ for the rewriting process. By the similar considerations for other compositions, the statement of the theorem holds.

It easy to see, that any $s \in S$ is a difference of $S \setminus \{s\}$ -reduced words. It means that $S$ is a reduced Gröbner–Shirshov basis. □

The above Gröbner–Shirshov basis of $B_4$ gives rise to the rewriting system (semi-Thue system) for $B_4$ that is defined by rules:

$$
\begin{align*}
    t_2^\pm t & \rightarrow t(t_2t_1^{-1})^\pm, \\
    b^\pm t & \rightarrow tt^\pm, \\
    a^\pm t_1 & \rightarrow t_1 b^\pm, \\
    a^\pm t_2 & \rightarrow t_2(b^{-1}a)^\pm, \\
    t_1^\pm t_1^{-1} & \rightarrow t^{-1}(t_2t_1^{-1})^\pm, \\
    b^\pm t_1^{-1} & \rightarrow t_1 a^\pm, \\
    b^\pm t_2^{-1} & \rightarrow t_2^{-1}(ba)^\pm, \\
    xx^{-1} & \rightarrow 1,
\end{align*}
$$

where $x \in \{a, b, t_1, t_2, t\}$.

As a particular case, we get the rewriting system for the braid group $B_3$.

References


New rewriting system for the braid group $B_4$


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