Abstract. The first example of a closed orientable hyperbolic 3-manifold was constructed by F. L"{o}bell in 1931 from eight copies of the right-angled 14-hedron. We consider the family of hyperbolic polyhedra which generalize the Lambert cube and the L"{o}bell polyhedron. For polyhedra from this family we give trigonometric relations between essential dihedral angles and lengths and obtain volume formulae in various forms.

1. Introduction

The theory of polyhedra in non-Euclidean spaces is one of many areas of mathematics that received a powerful influence from E.B. Vinberg (see, for example, [1, 25, 36, 37, 38]). His results on existence of hyperbolic Coxeter groups and on volume formulae for a wide class of hyperbolic polyhedra gave new life to this theory and attracted many investigators to it.

In the present paper we obtain volume formulae for a family of almost right-angled polyhedra in Lobachevsky space $\mathbb{H}^3$, i.e. for polyhedra with only a few dihedral angles not equal to $\pi/2$. The interest in this class of polyhedra is motivated by two classical examples: the Lambert cube and the L"{o}bell polyhedron.

Both polyhedra can be considered as natural generalizations of a Euclidean right-angled cube. In the first case, the combinatorial structure of the cube is preserved while dihedral angles vary. More precisely, let $L(\alpha_1, \alpha_2, \alpha_3)$ be a cube with dihedral angles $\alpha_1$, $\alpha_2$ and $\alpha_3$ as presented in Fig. 1 while all other dihedral angles are equal to $\pi/2$. It is well-known that if each of the $\alpha_i$'s belongs to $(0, \pi/2)$ then $L(\alpha_1, \alpha_2, \alpha_3)$ can be realized as a bounded polyhedron in $\mathbb{H}^3$. Each such polyhedron is called a Lambert cube in honor of J.H. Lambert (1729–1777) who investigated hyperbolic quadrilaterals with three right angles arising as faces of $L(\alpha_1, \alpha_2, \alpha_3)$.

Figure 1. Lambert cube $L(\alpha_1, \alpha_2, \alpha_3)$. 

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In the second case, right dihedral angles are preserved while the combinatorial structure varies. More precisely, for each \( n \geq 5 \) one can consider a right-angled polyhedron \( R(n) \) with \( n \)-gonal top and bottom and \( 2n \) pentagons on the lateral surface (see Fig. 2, where \( R(5) \), which is a dodecahedron, and \( R(6) \) are presented). The right-angled 14-hedron \( R(6) \) is called the L"obell polyhedron in honor of F.R. L"obell (1893–1964).

![Figure 2. Polyhedra \( R(5) \) and \( R(6) \).](image)

We recall, that in 1890 Klein, inspired by a number of examples due to Clifford, formulated the problem of describing all compact, connected, Riemannian manifolds of constant curvature. Then Killing showed that these manifolds are of the form \( X/\Gamma \), where \( X \) is a complete simply-connected space of constant curvature and \( \Gamma \) is a co-compact discrete group of isometries of \( X \), acting without fixed points. He called them Clifford – Klein space forms.

Examples of three-dimensional Clifford – Klein space forms of negative curvature were unknown for a long time [11, p.269-270]: “In bezug auf die hyperbolische Geometrie wollen wir nur hervorheben, daß eine geschlossene dreidimensionale hyperbolische Räumform von endlichem Volumen bis jetzt nicht gefunden zu sein scheint.”

Answering in the affirmative a question on the existence of Clifford – Klein space forms of constant negative curvature, L"obell [12] in 1931 constructed the first example of a closed orientable hyperbolic 3-manifold by gluing together eight copies of the L"obell polyhedron \( R(6) \). Some modification of L"obell’s construction was done in [29].

L"obell’s construction was reformulated algebraically and an infinite series of finite sets \( L(n) \), \( n \geq 5 \), of closed hyperbolic 3-manifolds were constructed in [16, 19, 33]. Each manifold from \( L(n) \), called a L"obell manifold, is constructed from eight copies of the right-angled \((2n+2)\)-hedron \( R(n) \). In particular, \( L(6) \) contains the first example constructed by L"obell. Moreover, \( L(n) \) contains both orientable and non-orientable manifolds. Essentially, the approach from [16, 19, 33] gives a method for constructing a closed orientable hyperbolic 3-manifold from eight copies of a right-angled polyhedron in \( \mathbb{H}^3 \), based on a coloring of its faces in four colors. A similar method was independently suggested by Takahashi [32]. Isometry groups of L"obell manifolds of a special kind were described in [15, 19]. The volume formula for L"obell manifolds was obtained in [34].

Recently L"obell manifolds, as well as right-angled polyhedra, have become objects of extensive investigations [2, 7, 9, 13, 14, 18, 28, 31]. In particular, arithmetical properties of these manifolds were investigated in [2]. Upper and lower bounds for the complexity of L"obell manifolds were obtained in [13, 14]. An arrangement of right-angled hyperbolic polyhedra by volume was done in [9]. It turns out that the smallest volume is attained by
the regular right-angled dodecahedron $R(5)$ and the second – by the 14-hedron $R(6)$. Four-dimensional generalizations of L"obell’s construction were considered in [7]. Right-angled polyhedra arising as convex cores of quasi-Fuchsian groups were investigated in [18].

Computations of volumes of polyhedra in $H^3$ is a difficult problem and explicit formulae for volumes are obtained only for some classes of polyhedra. Initial results in this direction were obtained by N.I. Lobachevskij. Nice expressions for volumes of some classes of polyhedra were obtained by Vinberg [1, 36, 38] and Kellerhals [10] in terms of the Lobachevsky function $\Lambda(x)$.

The aim of the present paper is to obtain explicit formulae for volumes of an infinite family of remarkable polyhedra in $H^3$. In particular, we generalize the following previously obtained results: formulae for volumes of Lambert cubes which were described in [3, 10], the volume formula for non-compact L"obell type polyhedra which was found in [18], and the volume formula for the polyhedra $R(n)$ (and so, the volume formula for L"obell manifolds) in terms of $\Lambda(x)$ which was obtained in [34] (its particular case, for $n$ divisibly by 6, was independently obtained in [31]).

The method we use to obtain volume formulae is the following. First, we relate the essential dihedral angles and lengths of polyhedra by a number of trigonometric relations similar to the Sine and Tangent rules for a triangle, using one more parameter (called a principal parameter). Then we find an algebraic equation, depending on the combinatorial type of the polyhedra, to evaluate this principal parameter. Finally, we apply the Schl"afli variation formula to find an explicit expression for the volume. This fruitful approach has been successfully applied in our previous papers [3, 4, 5, 17, 20] to find volume formulae for simplest knot and link cone-manifolds and polyhedra.

The paper is organized as follows. In Section 2 we recall some basic facts about the vector model of $H^3$, the Rivin theorem on existence of polyhedra in $H^3$, and the Schl"afli formula. In Section 3 we investigate the hyperbolic Lambert cube $L(\alpha_1, \alpha_2, \alpha_3)$ obtaining trigonometric relations between dihedral angles and lengths in Propositions 3.1 and 3.2 and volume formulae in Theorem 3.5. In Section 4 we consider a hyperbolic polyhedron $L(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ with two vertices at infinity. Trigonometric relations between its dihedral angles $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and lengths of corresponding edges are obtained in Propositions 4.1, 4.2, and 4.3. These relations are used in Theorem 4.6 to obtain volume formulae. In Section 5 we consider an infinite family of almost right-angled hyperbolic polyhedra generalizing $R(n)$. Using a trigonometric relation between dihedral angles and lengths (see Proposition 5.1) we obtain volume formula for these polyhedra in Theorem 5.2.

The authors are thankful to Professor E.B. Vinberg for bringing to their attention L"obell’s paper [12] in 1977. This paper became a source of new ideas and results during the last thirty years.

2. Preliminary facts

2.1. The model of the hyperbolic 3-space. We will use the following vector model of hyperbolic 3-space $H^3$ [1, 26]. A scalar product in $\mathbb{R}^4$ defined for vectors $x = (x_0, x_1, x_2, x_3)$ and $y = (y_0, y_1, y_2, y_3)$ by the formula

$$
(x, y) = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3,
$$

$$
3
$$
turns $\mathbb{R}^4$ into a pseudo-Euclidean space $\mathbb{R}^{3,1}$. A basis $\{e_0, e_1, e_2, e_3\}$ in $\mathbb{R}^{3,1}$ is said to be orthonormal if $(e_0, e_0) = -1$ and $(e_i, e_i) = 1$ for $i = 1, 2, 3$, and $(e_i, e_j) = 0$ for $i \neq j$, $i, j \in \{0, 1, 2, 3\}$.

Consider

$$\mathbb{H}^3 = \{x \in \mathbb{R}^{3,1} : (x, x) = -1, \quad x_0 > 0\}$$

with the Riemannian metric induced by the pseudo-Euclidean metric on $\mathbb{R}^{3,1}$:

$$ds^2 = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2.$$ 

Any hyperplane $H \subset \mathbb{H}^3$ is of the form $H_e = \{x \in \mathbb{H}^3 : (x, e) = 0\}$, where $e \in \mathbb{R}^{3,1}$ and $(e, e) = 1$. Let $H_e^- = \{x \in \mathbb{H}^3 : (x, e) \leq 0\}$ be one of two subspaces determined by $H_e$. Two hyperplanes $H_e$ and $H_f$ are intersecting if $|(e, f)| < 1$, and dihedral angle $\alpha_{e,f}$ between $H_e$ and $H_f^-$ is given by $\cos \alpha_{e,f} = -(e, f)$. Two hyperplanes $H_e$ and $H_f$ are parallel if $|(e, f)| = 1$. Two hyperplanes $H_e$ and $H_f$ are divergent if $|(e, f)| > 1$, and distance $\rho_{e,f}$ between them (realized by a geodesic in $H_e^- \cap H_f^-$) is given by $\cosh \rho_{e,f} = -(e, f)$.

A convex polyhedron $P \subset \mathbb{H}^3$ is defined as an intersection of finitely many half-spaces:

$$P = \bigcap_{i=1}^n H_i^-,$$

where each hyperplane $H_i$ is determined by its normal vector $e_i$ as above. It may always be assumed that none of half-spaces $H_i^-$ contains the intersection of all others. Under this condition half-spaces $H_i^-$ are uniquely determined by $P$. The Gram matrix of the system of vectors $\{e_1, \ldots, e_s\}$ is said to be Gram matrix of the polyhedron $P$.

2.2. The Rivin theorem. To demonstrate existence of polyhedra in $\mathbb{H}^3$ we will use the following characterization due to Rivin [8], that is done in terms of a Gaussian image of a polyhedron.

Let $P$ be a compact convex polyhedron in $\mathbb{H}^3$. The Gauss map $\mathcal{G}$ from the boundary $\partial P$ of $P$ to the unit sphere $S^2$ is a set-valued function which assigns to each point $p \in \partial P$ the set of outward unit normals to planes supporting $\partial P$ at $p$. Thus, the whole of a face $f$ of $\partial P$ is mapped under $\mathcal{G}$ to a single point which is outward unit normal to $f$. An edge $e$ of $\partial P$ is mapped to a geodesic segment $\mathcal{G}(e)$ on $S^2$, whose length is easily seen to be the exterior dihedral angle at the edge $e$. A vertex $v$ of $\partial P$ is mapped by $\mathcal{G}$ to a spherical polygon $\mathcal{G}(v)$, whose sides are the images under $\mathcal{G}$ of edges incident to $v$ and whose angles are easily seen to be the angles supplementary to the planar angles of the faces incident to $v$; that is, $\mathcal{G}(e_1)$ and $\mathcal{G}(e_2)$ meet at angle $\pi - \alpha$ whenever $e_1$ and $e_2$ meet at angle $\alpha$. Glue the resulting polygons together into a closed surface, using the rule that faces $\mathcal{G}(v_1)$ and $\mathcal{G}(v_2)$ are gluing isometrically whenever vertices $v_1$ and $v_2$ share an edge.

The resulting metric space $\mathcal{G}(\partial P)$ is topologically the sphere $S^2$ and the complex is combinatorially dual to $\partial P$. But metrically it is a cone-manifold with the underlying space $S^2$. In general case the sum of angles of a bounded hyperbolic $n$-gon $f$ is less than $2\pi(n - 2)$. Hence the sum of angles around a vertex $\mathcal{G}(f)$ is greater than $2\pi$. Thus vertices $\mathcal{G}(f)$ of $\mathcal{G}(\partial P)$ are cone points with cone angle greater than $2\pi$.

The following theorem gives a precise characterization of those cone-manifolds that can arise as an image $\mathcal{G}(\partial P)$ for a compact convex polyhedron $P$ in $\mathbb{H}^3$. 


Theorem 2.1. [8] A metric space \((M, g)\) homeomorphic to \(S^2\) can arise as the Gaussian image \(G(\partial P)\) of boundary of a compact convex polyhedron \(P\) in \(H^3\) if and only if the following conditions hold:

(a) The metric \(g\) has constant curvature 1 away from a finite collection of cone points \(c_i\).
(b) The cone angles at \(c_i\) are greater than \(2\pi\).
(c) The lengths of closed geodesics of \((M, g)\) are all strictly greater than \(2\pi\).

Furthermore the metric of \(G(\partial P)\) determines the hyperbolic polyhedron \(P\) uniquely (up to a motion).

2.3. The Schlafli formula. To obtain volume formulae we will use the following Schlafli variation formula (see [1, p. 119] for proof and more discussions).

Theorem 2.2. Let a convex polyhedron in \(H^n\) be deformed in such a way that its combinatorial structure is preserved while its dihedral angles vary in a differentiable manner. Under these conditions its volume is also varying in a differentiable manner, and its differential is given by the Schlafli formula

\[
d\text{vol} P = -\frac{1}{n-1} \sum_{\text{dim}F = n-2} \text{vol} F \, d\alpha_F,
\]

where the sum is taken over all \((n-2)\)-dimensional faces \(F\) of the polyhedron \(P\) and \(\alpha_F\) denotes the dihedral angle at the face.

In the case of polyhedra in \(H^3\) the Schlafli formula reduces to

\[
d\text{vol} P = -\frac{1}{2} \sum_i \ell_i \, d\alpha_i,
\]

where the sum is taken over all edges of \(P\), and \(\ell_i\) denotes the length of the \(i\)-th edge and \(\alpha_i\) is the dihedral angle at this edge.

3. The Lambert cube \(L(\alpha_1, \alpha_2, \alpha_3)\)

3.1. The existence. Consider the Lambert cube \(L(\alpha_1, \alpha_2, \alpha_3)\) presented by its projection in Fig. 3 with three outgoing edges meeting in a point. Combinatorially it is a cube. Three of its dihedral angles are essential and equal to \(\alpha_1, \alpha_2, \alpha_3\), while all other dihedral angles are suppose to be \(\pi/2\). Combinatorially, the Gaussian image \(G(\partial L(\alpha_1, \alpha_2, \alpha_3))\) is an octahedron. Three its edges are of lengths \(\bar{\alpha}_i = \pi - \alpha_i\), \(i = 1, 2, 3\), while all other edges are of length \(\pi/2\). It has six cone points: for each \(i\) there are two cone points with the cone angle \(3\pi/2 + (\pi - \alpha_i) = 2\pi + (\pi/2 - \alpha_i)\). Thus, to obtain cone angles greater than \(2\pi\) we need to assume that \(\alpha_i < \pi/2\) for all \(i\). Obviously, these inequalities also guarantee that lengths of any closed geodesic is greater than \(2\pi\). Indeed, any closed geodesic consists of at least four segments with length \(\geq \pi/2\) with the strong inequality for at least one of segments. Therefore, by Theorem 2.1 the Lambert cube \(L(\alpha_1, \alpha_2, \alpha_3)\) is a bounded hyperbolic polyhedron if and only if all its essential angles are less than \(\pi/2\). (Recall that \(L(\pi/2, \pi/2, \pi/2)\) is the Euclidean cube.)
3.2. Relations between angles and lengths. Let us denote by $\ell_1$, $\ell_2$, and $\ell_3$ the lengths of the edges of the hyperbolic Lambert cube $\mathcal{L}(\alpha_1, \alpha_2, \alpha_3)$ corresponding to the essential angles $\alpha_1$, $\alpha_2$, and $\alpha_3$ respectively. We call $\ell_1$, $\ell_2$, and $\ell_3$ essential lengths. The following proposition gives relations between these essential parameters.

**Proposition 3.1** (Sine-cosine Rule). The essential angles and lengths of the hyperbolic Lambert cube $\mathcal{L}(\alpha_1, \alpha_2, \alpha_3)$ satisfy the following relation:

\[
\frac{\sin \alpha_i}{\sinh \ell_i} \cdot \frac{\sin \alpha_{i+1}}{\sinh \ell_{i+1}} \cdot \frac{\cos \alpha_{i+2}}{\cosh \ell_{i+2}} = 1,
\]

where $i = 1, 2, 3$ and all suffices are taken by mod 3.

**Proof.** Denote by $e_1$, $e_2$, $e_3$, $f_1$, $f_2$, and $f_3$ normal vectors in $\mathbb{R}^{3,1}$ determine hyperplanes $H_{e_1}$, $H_{e_2}$, $H_{e_3}$, $H_{f_1}$, $H_{f_2}$, $H_{f_3}$ supporting faces of $\mathcal{L}(\alpha_1, \alpha_2, \alpha_3)$ (see Fig. 3). Assume that vectors are oriented in such a way that

\[
\mathcal{L}(\alpha_1, \alpha_2, \alpha_3) = H_{e_1} \cap H_{e_2} \cap H_{e_3} \cap H_{f_1} \cap H_{f_2} \cap H_{f_3}.
\]

By the definition of $\mathcal{L}(\alpha_1, \alpha_2, \alpha_3)$ we have

\[
(e_i, e_i) = 1, \quad (f_i, f_i) = 1, \quad (e_i, e_{i+1}) = 0, \quad (f_i, f_{i+1}) = 0,
\]

\[
(e_i, f_i) = -\cosh \ell_i, \quad (e_{i+1}, f_i) = -\cos \alpha_{i+2},
\]

for $i = 1, 2, 3$, where all suffices are taken by mod 3. As one can see from Fig. 3, faces determined by vectors $e_1$, $e_2$, and $e_3$ are mutually orthogonal and are intersecting in one point that is a vertex of the cube. Let $e_0 \in \mathbb{R}^{3,1}$ be the vector presenting this vertex. Since $e_0 \in H_{e_1} \cap H_{e_2} \cap H_{e_3}$ we get $(e_i, e_0) = 0$ for $i = 1, 2, 3$. Hence

\[
(e_i, e_j) = 0, \quad 0 \leq i < j \leq 3; \quad (e_i, e_i) = 1, \quad i = 1, 2, 3; \quad (e_0, e_0) = -1,
\]

and vectors \(\{e_0, e_1, e_2, e_3\}\) are forming an orthonormal basis in $\mathbb{R}^{3,1}$. Hence, vectors $f_1$, $f_2$, and $f_3$ can be expressed as linear combinations of these basis vectors.

Suppose that $f_1$ is a linear combination of the form

\[
f_1 = d \cdot e_0 + a \cdot e_1 + b \cdot e_2 + c \cdot e_3
\]

![Figure 3. $\mathcal{L}(\alpha_1, \alpha_2, \alpha_3)$ and $\mathcal{G}(\partial \mathcal{L}(\alpha_1, \alpha_2, \alpha_3))$.](image)
for some $a, b, c, d \in \mathbb{R}$. Then

$$a = (f_1, e_1) = -\cosh \ell_1, \quad b = (f_1, e_2) = -\cos \alpha_3, \quad c = (f_1, e_3) = 0,$$

and

$$1 = (f_1, f_1) = -a^2 + b^2 + c^2.$$

Therefore

$$d^2 = a^2 + b^2 - 1 = \cosh^2 \ell_1 + \cos^2 \alpha_3 - 1 = \cosh^2 \ell_1 - \sin^2 \alpha_3.$$

Since $e_0 \in H_{f_1}$, we have $d = -(f_1, e_0) > 0$, and $d = \sqrt{\cosh^2 \ell_1 - \sin^2 \alpha_3}$.

Applying similar considerations for $f_2$ and $f_3$ we obtain:

$$\begin{align*}
    f_1 &= d_1 \cdot e_0 - \cosh \ell_1 \cdot e_1 - \cos \alpha_3 \cdot e_2 + 0 \cdot e_3 \\
    f_2 &= d_2 \cdot e_0 + 0 \cdot e_1 - \cosh \ell_2 \cdot e_2 - \cos \alpha_3 \cdot e_3 \\
    f_3 &= d_3 \cdot e_0 - \cos \alpha_2 \cdot e_1 + 0 \cdot e_2 - \cosh \ell_3 \cdot e_3
\end{align*}$$

with $d_i$ given by

$$d_i = \sqrt{\cosh^2 \ell_i - \sin^2 \alpha_{i+2}}.$$

For $i = 1, 2, 3$ from $(f_i, f_{i+1}) = 0$ we get

$$d_i d_{i+1} = \cos \alpha_{i+2} \cosh \ell_{i+1}.$$

Therefore

$$d_i^2 d_{i+1}^2 = \cos^2 \alpha_{i+2} \cosh^2 \ell_{i+1},$$

and using $d_{i+1}^2 = \cosh^2 \ell_{i+1} - \sin^2 \alpha_i$, we get

$$d_i^2 \cosh^2 \ell_{i+1} - d_i^2 \sin^2 \alpha_i = \cos^2 \alpha_{i+2} \cosh^2 \ell_{i+1}.$$

Hence

$$(d_i^2 - \cos^2 \alpha_{i+2}) \cosh^2 \ell_{i+1} = d_i^2 \sin^2 \alpha_i.$$

Using $d_i^2 = \cosh^2 \ell_i$, we get

$$\sinh \ell_i \cosh \ell_{i+1} = d_i \sin \alpha_i,$$

whence

$$d_i = \frac{\sinh \ell_i \cosh \ell_{i+1}}{\sin \alpha_i}.$$

Substituting expressions for $d_i$ and $d_{i+1}$ into (3) we have

$$\frac{\sinh \ell_i \cosh \ell_{i+1}}{\sin \alpha_i} \cdot \frac{\sinh \ell_{i+1} \cosh \ell_{i+2}}{\sinh \alpha_{i+1}} = \cos \alpha_{i+2} \cosh \ell_{i+1},$$

that implies the formula (2).

□

**Proposition 3.2** (Tangent Rule). The essential angles and lengths of the hyperbolic Lambert cube $L(\alpha_1, \alpha_2, \alpha_3)$ satisfy the following relations:

$$\tan \alpha_1 \tanh \ell_1 = \tan \alpha_2 \tanh \ell_2 = \tan \alpha_3 \tanh \ell_3 = T,$$

where $T$ is a positive root of the equation

$$T^2 = \prod_{i=1}^{3} \frac{T^2 - \tan^2 \alpha_i}{1 + \tan^2 \alpha_i}.$$
Proof. Formula (2) implies
\[
\frac{\sin \alpha_i \sin \alpha_{i+1}}{\sinh \ell_i \sinh \ell_{i+1}} = \frac{\sin \alpha_{i+2}}{\sinh \ell_{i+2}} = \frac{\tan \alpha_{i+2}}{\tanh \ell_{i+2}}
\]
for \(i = 1, 2, 3\) with all suffices taken by mod 3. Thus, we have three equations with the same expression on the left. Denote it by \(T\):
\[
T = \frac{\tan \alpha_1}{\tanh \ell_1} = \frac{\tan \alpha_2}{\tanh \ell_2} = \frac{\tan \alpha_3}{\tanh \ell_3}.
\]
Relation (5) follows from an elementary observation:
\[
\frac{\sin^2 \alpha_i}{\sinh^2 \ell_i} = \frac{T^2 - \tan^2 \alpha_i}{1 + \tan^2 \alpha_i}.
\]

\[\square\]

Remark 3.3. To simplify notations, let us denote \(A_i = \tan \alpha_i\) for \(i = 1, 2, 3\). It is easy to see that (5) is equivalent to the equation \((T^2 + 1)P(T^2) = 0\), where
\[
P(x) = x^2 - \left(1 + \sum_{i=1}^{3} A_i^2\right)x - \prod_{i=1}^{3} A_i^2.
\]
Hence, the principal parameter \(T\) from Proposition 3.2 is the positive root of the equation
\[
T^2 = K + \left(K^2 + \prod_{i=1}^{3} A_i^2\right)^{1/2},
\]
where \(K = (1 + \sum_{i=1}^{3} A_i^2)/2\). We observe that \(T\) is the largest real root of the equation \(P(x^2) = 0\) and \(P(x^2)\) is positive if \(x^2\) tends to infinity. Hence \(P(x^2) > 0\) for any \(x > T\). Straightforward calculations shows that
\[
P(\tan^2 \alpha_i) = -\tan^2 \alpha_i(1 + \tan^2 \alpha_{i-1})(1 + \tan^2 \alpha_{i+1}) < 0,
\]
that implies \(T > \tan \alpha_i\) for \(i = 1, 2, 3\).

3.3. The volume formulae. Recall [1, 26] that volumes of polyhedra in \(\mathbb{H}^3\) traditionally are expressed in terms of the Lobachevsky function
\[
\Lambda(x) = -\int_0^x \log |2 \sin \zeta| \, d\zeta.
\]
Let us introduce the function
\[
\Delta(\lambda, \sigma) = \Lambda(\lambda + \sigma) - \Lambda(\lambda - \sigma)
\]
which admits to simplify volume formulae in many cases due to the following Lemma.

Lemma 3.4. [18] Consider
\[
I(L, S) = \int_S^{+\infty} \log \left| \frac{t^2 - L^2}{1 + t^2} \right| \frac{dt}{1 + t^2},
\]
where \(L = \tan \lambda, S = \tan \sigma,\) and \(0 < \lambda, \sigma < \frac{\pi}{2}\). Then
\[
I(L, S) = \Delta(\lambda, \sigma) - \Delta(\pi/2, \sigma).
\]
Theorem 3.5. The volume of the hyperbolic Lambert cube \( \mathcal{L}(\alpha_1, \alpha_2, \alpha_3) \), \( 0 < \alpha_1, \alpha_2, \alpha_3 < \pi/2 \), is given by formulae (6) and (7):

\[
(6) \quad \text{vol}\, \mathcal{L}(\alpha_1, \alpha_2, \alpha_3) = \frac{1}{4} \int_{T}^{\infty} \log \left[ \frac{1}{t^2} \prod_{i=1}^{3} \frac{t^2 - \tan^2 \alpha_i}{1 + \tan^2 \alpha_i} \right] \frac{dt}{1 + t^2},
\]

where \( T \) is a positive root of the equation

\[
T^4 - \left( 1 + \sum_{i=1}^{3} \tan^2 \alpha_i \right) T^2 - \prod_{i=1}^{3} \tan^2 \alpha_i = 0;
\]

\[
(7) \quad \text{vol}\, \mathcal{L}(\alpha_1, \alpha_2, \alpha_3) = \frac{1}{4} \left[ \sum_{i=1}^{3} \Delta(\alpha_i, \theta) - \Delta(0, \theta) - 2\Delta(\pi/2, \theta) \right],
\]

where \( \theta, 0 < \theta < \pi/2 \), is given by \( \tan \theta = T \).

Proof. Let \( V = V(\alpha_1, \alpha_2, \alpha_3) = \text{vol}\, \mathcal{L}(\alpha_1, \alpha_2, \alpha_3) \) be the volume function. To find \( V \) we are going to use the Schlafli formula (1) and then to integrate the differential form

\[
\omega = dV = -\frac{1}{2} \sum_{i=1}^{3} \ell_i d\alpha_i
\]

which is defined for \( 0 < \alpha_1, \alpha_2, \alpha_3 < \pi/2 \). To simplify notations, denote \( A_i = \tan \alpha_i \). Then \( d\alpha_i = dA_i/(1 + A_i^2) \), and by (4) we have \( \ell_i = \arctanh (A_i/T) \). Thus we need to integrate the differential form

\[
\omega = -\frac{1}{2} \sum_{i=1}^{3} \arctanh (A_i/T) \frac{dA_i}{1 + A_i^2}
\]

with \( T \) satisfying (5). In order to do this we will find an extended differential form \( \Omega \) of four independent variables \( T, A_1, A_2, \) and \( A_3 \):

\[
\Omega = -\frac{1}{2} \left( \sum_{i=1}^{3} \arctanh (A_i/T) \frac{dA_i}{1 + A_i^2} + \ell(A_1, A_2, A_3, T) \frac{dT}{1 + T^2} \right)
\]

with the following properties:

(1) \( \Omega \) is smooth and exact in the region

\[
\mathcal{U} = \{(A_1, A_2, A_3, T) \in \mathbb{R}^4 : T > A_i > 0, \ i = 1, 2, 3 \};
\]

(2) \( \Omega = \omega \) for all \( (A_1, A_2, A_3, T) \in \mathcal{U} \) satisfying the equation (5).

Let \( \ell = \ell(A_1, A_2, A_3, T) \). Since \( \Omega \) is supposed to be exact we have

\[
\frac{1}{1 + T^2} \frac{\partial \ell}{\partial A_i} = \frac{1}{1 + A_i^2} \frac{\partial}{\partial T} (\arctanh (A_i/T)) = \frac{A_i}{(1 + A_i^2)(A_i^2 - T^2)}.
\]

Hence

\[
\ell = \int \frac{(1 + T^2)A_i}{(1 + A_i^2)(A_i^2 - T^2)} \, dA_i = \frac{1}{2} \log \frac{T^2 - A_i^2}{1 + A_i^2} + C,
\]

where \( C \) depends on the variables \( T, A_j, j \neq i \). Since the considerations are symmetric with respect to \( A_1, A_2 \) and \( A_3 \) we obtain

\[
\ell = \frac{1}{2} \log \left[ \prod_{i=1}^{3} \frac{T^2 - A_i^2}{1 + A_i^2} \right] + \phi(T),
\]

where \( \phi(T) \) depends on \( T \) only.
where the function \( C = \phi(T) \) depends only. The choice \( \phi(T) = -\log T \) assures that condition (20) will be satisfied. So, we have

\[
\ell = \ell(A_1, A_2, A_3, T) = \frac{1}{2} \log \left[ \frac{1}{T^2} \prod_{i=1}^{3} \frac{T^2 - A_i^2}{1 + A_i^2} \right].
\]

Now we are able to prove formula (6). Consider the function

\[
W = W(A_1, A_2, A_3, T) = \frac{1}{4} \int_{T}^{\infty} \log \left[ \frac{1}{t^2} \prod_{i=1}^{3} \frac{t^2 - A_i^2}{1 + A_i^2} \right] \frac{dt}{1 + t^2},
\]

where \( T \) is a positive root of (5). By Remark 3.3 for \( t \geq T \) we have \( t^2 - A_i^2 \geq T^2 - A_i^2 > 0 \) for \( i = 1, 2, 3 \). Hence, the expression under the log in the above integral is positive.

Straightforward calculations show that

\[
\frac{\partial W}{\partial A_i} = -\frac{\arctanh (A_i/T)}{2(1 + A_i^2)}
\]

for \( i = 1, 2, 3 \) and \( W(A_1, A_2, A_3, T) \to 0 \) as \( A_1, A_2, A_3 \to \infty \). By (1) and Proposition 3.2 the volume function \( V = V(\alpha_1, \alpha_2, \alpha_3) = V(A_1, A_2, A_3) \) satisfies the same conditions:

\[
\frac{\partial V}{\partial A_i} = \frac{\partial V}{\partial \alpha_i} \frac{d\alpha_i}{dA_i} = -\frac{\ell_i}{2} \frac{1}{1 + A_i^2} = -\frac{\arctanh (A_i/T)}{2(1 + A_i^2)}
\]

for \( i = 1, 2, 3 \) and \( V(A_1, A_2, A_3) \to 0 \) as \( A_1, A_2, A_3 \to \infty \). Hence the functions \( V(A_1, A_2, A_3) \) and \( W(A_1, A_2, A_3, T) \) coincide and (6) holds.

The formula (7) follows immediately from (6) by Lemma 3.4.

\[\square\]

**Remark 3.6.** Propositions 3.1 and 3.2 were already observed by Pashkevich in [24], where taking \( 1/T \) as a principal parameter the analog of (6) was obtained. Formula (7) was originally obtained by Mednykh [17]. This is a refined version of Kellerhals formula [10] for the volume of the hyperbolic Lambert cube in terms of the Lobachevsky function. In [10] the parameter \( \theta \) was calculated in a slightly more complicated way by making use of dihedral angles and lengths of polyhedron. For the spherical version of this result see [3].

4. **The polyhedron** \( \mathcal{L}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \)

4.1. **The existence.** Consider a polyhedron \( \mathcal{L}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \) presented by its projection in Fig. 4 with four outgoing edges meeting in a point. This polyhedron has two 4-valent vertices and eight 3-valent vertices, and all its faces are quadrilaterals. If \( 0 < \alpha_1, \alpha_2, \alpha_3, \alpha_4 < \pi \) then by [27] \( \mathcal{L}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \) can be realized in \( \mathbb{H}^3 \) in such a way that 4-valent vertices are ideal points; 3-valent vertices are finite points; four essential dihedral angles are equal to \( \alpha_1, \alpha_2, \alpha_3, \) and \( \alpha_4, \) while all other dihedral angles are \( \pi/2. \)

We point out that the polyhedron \( \mathcal{L}(\alpha_1, \pi/2, \alpha_3, \pi/2) \) arose in [18] as the convex core of a quasi-Fuchsian punctured torus group, where some relations between dihedral angles and lengths of the polyhedron and the volume of the polyhedron were obtained for this particular case.
Figure 4. The polyhedron $L(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$.

4.2. Relations between angles and lengths. Let us denote by $\ell_1$, $\ell_2$, $\ell_3$, and $\ell_4$ the lengths of the edges corresponding to the essential dihedral angles $\alpha_1$, $\alpha_2$, $\alpha_3$, and $\alpha_4$ respectively. We call these lengths essential. In this subsection we will find relations between the essential angles and lengths of the polyhedron.

Consider two adjacent quadrilaterals of $L(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ pictured in Fig. 5. One has angles $\alpha_i$, 0, $\pi/2$, $\pi/2$ and another – angles $\alpha_{i+1}$, 0, $\pi/2$, $\pi/2$. Denote the length of the common edge by $x_i$.

From the equations for hyperbolic quadrilaterals [6] we have

$$\begin{cases}
\cos 0 = - \cos \alpha_{i+1} \cosh \ell_i + \sin \alpha_{i+1} \sinh \ell_i \sinh x_i, \\
\cos 0 = - \cos \alpha_i \cosh \ell_{i+1} + \sin \alpha_i \sinh \ell_{i+1} \sinh x_i.
\end{cases}$$

Hence

$$\frac{\sin \alpha_{i+1} \sinh \ell_i}{1 + \cos \alpha_{i+1} \cosh \ell_i} = \frac{\sin \alpha_i \sinh \ell_{i+1}}{1 + \cos \alpha_i \cosh \ell_{i+1}}.$$ 

Using the identity

$$(\sin \alpha \sinh \ell)^2 + (1 + \cos \alpha \cosh \ell)^2 = (\cos \alpha + \cosh \ell)^2,$$
we get
\[(8) \quad \frac{\cos \alpha_{i+1} + \cosh \ell_i}{1 + \cos \alpha_{i+1} \cosh \ell_i} = \frac{\cos \alpha_i + \cosh \ell_{i+1}}{1 + \cos \alpha_i \cosh \ell_{i+1}}.\]

Note that \(\cos \alpha_{i+1} + \cosh \ell_i > \cos \alpha_{i+1} + 1 > 0\) and
\[1 + \cos \alpha_{i+1} \cosh \ell_i = \sin \alpha_{i+1} \sinh \ell_i \sinh x_i > 0.\]

Analogously, \(\cos \alpha_i + \cosh \ell_{i+1} > 0\) and \(1 + \cos \alpha_i \cosh \ell_{i+1} > 0\). Rearranging (8) we obtain
\[(9) \quad v = \frac{1 - \cosh \ell_i \cos \alpha_i}{\cosh \ell_i - \cos \alpha_i},\]

we get
\[(10) \quad \cosh \ell_i = \frac{1 + v \cos \alpha_i}{v + \cos \alpha_i}\]

and
\[\cos \alpha_i = \frac{1 - v \cosh \ell_i}{\cosh \ell_i - v}.\]

Suppose that the faces of \(L(\alpha_1, \alpha_2, \alpha_3, \alpha_4)\) are numbered as in Fig. 4. Denote by \(\rho(i, j)\) the distance between faces numbered by \(i\) and \(j\). Let us introduce notations \(v = \cosh \rho(1, 7), w = \cosh \rho(2, 8), t = \cosh \rho(3, 5), s = \cosh \rho(4, 6)\). Let \(G\) be the Gram matrix of the polyhedron \(L(\alpha_1, \alpha_2, \alpha_3, \alpha_4)\), i.e. the \(8 \times 8\)-matrix:

\[
\begin{pmatrix}
1 & 0 & -1 & 0 & 0 & -\cos \alpha_4 & -v & -\cosh \ell_1 \\
0 & 1 & 0 & -1 & -\cosh \ell_4 & 0 & -\cos \alpha_3 & -w \\
-1 & 0 & 1 & 0 & -t & -\cosh \ell_3 & 0 & -\cos \alpha_2 \\
0 & -1 & 0 & 1 & -\cos \alpha_4 & -s & -\cosh \ell_4 & 0 \\
0 & -\cosh \ell_4 & -t & -\cos \alpha_1 & 1 & 0 & -1 & 0 \\
-\cos \alpha_4 & 0 & -\cos \alpha_3 & -s & 0 & 1 & 0 & 1 \\
-\cosh \ell_4 & -w & -\cosh \ell_2 & -1 & 0 & 1 & 0 & 0 \\
-\cosh \ell_4 & -w & -\cos \alpha_2 & 0 & 0 & -1 & 0 & 1
\end{pmatrix}
\]

By [1] the rank of \(G\) is at most four. Denote by \(G(i_1, i_2, i_3, i_4, i_5)\), the main minor of \(G\), formed by lines and columns with numbers \(i_1, i_2, i_3, i_4, i_5\). The vanishing of the determinants of the \(5 \times 5\)-minors of \(G\) gives us relations between the entries of \(G\). Taking the minors corresponding to the columns \((1, 2, 4, 5, 6), (1, 2, 3, 5, 6), (3, 4, 5, 7, 8), (3, 4, 6, 7, 8)\), respectively, we obtain following four equations:

\[(11) \quad s^2 (\cosh^2 \ell_4 - 1) = (\cosh \ell_4 + \cos \alpha_4)^2 (1 - \cos^2 \alpha_4),\]
\[(12) \quad t^2 (1 - \cos^2 \alpha_4) = (\cosh \ell_3 + \cos \alpha_2)^2 (\cosh^2 \ell_4 - 1),\]
\[(13) \quad t^2 (\cosh^2 \ell_2 - 1) = (\cosh \ell_2 + \cos \alpha_1)^2 (1 - \cos^2 \alpha_2),\]
\[(14) \quad s^2 (1 - \cos^2 \alpha_2) = (\cosh \ell_3 + \cos \alpha_2)^2 (\cosh^2 \ell_2 - 1).\]
Compare the product of left sides of (11) and (12) with the product of right sides of the same equations and, analogously, compare the product of left sides of (13) and (14) with the product of right sides of the same equations:

$$\begin{align*}
&\begin{cases}
  s t = (\cosh \ell_3 + \cos \alpha_4)(\cosh \ell_4 + \cos \alpha_1), \\
  s t = (\cosh \ell_2 + \cos \alpha_1)(\cosh \ell_3 + \cos \alpha_2).
\end{cases}
\end{align*}$$

Hence

(15) \((\cosh \ell_3 + \cos \alpha_4)(\cosh \ell_4 + \cos \alpha_1) = (\cosh \ell_2 + \cos \alpha_1)(\cosh \ell_3 + \cos \alpha_2)\)

Substituting expressions (10) for \(\ell_i\) in terms of \(v\) and \(\alpha_i\), we have

$$\left(\frac{1 + v \cos \alpha_3}{v + \cos \alpha_3} + \cos \alpha_4\right) \left(\frac{1 + v \cos \alpha_4}{v + \cos \alpha_4} + \cos \alpha_1\right) = \left(\frac{1 + v \cos \alpha_2}{v + \cos \alpha_2} + \cos \alpha_1\right) \left(\frac{1 + v \cos \alpha_3}{v + \cos \alpha_3} + \cos \alpha_2\right).$$

Hence

(16) \((1 - v^2)^2 = \prod_{i=1}^{4} (v + \cos \alpha_i)\).

Denote \(\overline{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)\). Since \(\cosh \ell_i - \cos \alpha_i > 0\), the function \(v = v(\overline{\alpha})\) defined by (9) is continuous in the region \(U = \{\overline{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) : 0 < \alpha_i < \pi, i = 1, \ldots, 4\}\).

Note that if \(\overline{\alpha} \in \mathcal{U}\) then \(v(\overline{\alpha}) \neq 0\) and \(v(\overline{\alpha}) \neq 1\). Indeed, if \(v(\alpha) = 0\) then from (16) we get \(1 = \prod_{i=1}^{4} \cos \alpha_i\) and so \(\overline{\alpha} \notin \mathcal{U}\). Similar, if \(v(\overline{\alpha}) = 1\) then from (16) we get 0 = \(\prod_{i=1}^{4} (1+ \cos \alpha_i)\) and again \(\overline{\alpha} \notin \mathcal{U}\). For particular value \(\overline{\alpha} = (\pi/2, \pi/2, \pi/2, \pi/2)\) we get \(v(\overline{\alpha}) = 1/ \cosh \ell_i < 1\) and \((1 - v(\overline{\alpha})^2)^2 = 1\), whence \(v(\overline{\alpha}) = 1/\sqrt{2} \in (0; 1)\). By continuity, we conclude that \(v(\overline{\alpha}) \in (0; 1)\) for all \(\overline{\alpha} \in \mathcal{U}\).

Thus we have

**Proposition 4.1.** The essential angles and lengths of the hyperbolic polyhedron \(\mathcal{L}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)\) satisfy the following relation for all \(i = 1, \ldots, 4\):

(17) \(1 - \cosh \ell_i \cos \alpha_i / \cosh \ell_i - \cos \alpha_i = v\),

where \(v\), \(0 < v < 1\), is a root of the equation (16).

Taking into account a new principal parameter

\(T^2 = \frac{1 + v}{1 - v} = \frac{\tan^2(\alpha_i/2)}{\tanh^2(\ell_i/2)}\),

we obtain the following two propositions.

**Proposition 4.2** (Tangent Rule). The essential angles and lengths of the hyperbolic polyhedron \(\mathcal{L}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)\) satisfy the following relations:

(18) \(\frac{\tan(\alpha_1/2)}{\tanh(\ell_1/2)} = \frac{\tan(\alpha_2/2)}{\tanh(\ell_2/2)} = \frac{\tan(\alpha_3/2)}{\tanh(\ell_3/2)} = \frac{\tan(\alpha_4/2)}{\tanh(\ell_4/2)} = T\).
where, $T, T > 1$, is a root of the equation

$$T^4 = \prod_{i=1}^{4} \frac{T^2 - \tan^2(\alpha_i/2)}{1 + \tan^2(\alpha_i/2)}.$$  

**Proposition 4.3** (Sine Rule). The essential angles and lengths of the hyperbolic polyhedron $L(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ satisfy the following relation:

$$\frac{\sin \alpha_1}{\sinh \ell_1} \cdot \frac{\sin \alpha_2}{\sinh \ell_2} \cdot \frac{\sin \alpha_3}{\sinh \ell_3} \cdot \frac{\sin \alpha_4}{\sinh \ell_4} = 1. \tag{20}$$

**Proof.** Using (18) we have

$$\frac{T^2 - \tan^2(\alpha_i/2)}{T(1 + \tan^2(\alpha_i/2))} = \frac{\sin \alpha_i}{\sinh \ell_i}.$$ 

Hence (19) can be rewritten in the form (20). \hfill \Box

**Remark 4.4.** To simplify notations, let us denote $A_i = \tan(\alpha_i/2)$ for $i = 1, \ldots, 4$. The equation (19) is equivalent to the equation $(T^2 + 1)Q(T^2) = 0$, where $Q(x)$ is a cubic polynomial

$$Q(x) = x^3 - \left(1 + \frac{4}{1} A_1^2\right) x^2 - \left(\sum_{1 \leq i < j < k \leq 4} A_i^2 A_j^2 A_k^2 + \prod_{i=1}^{4} A_i^2\right) x + \prod_{i=1}^{4} A_i^2.$$ 

Since $Q(0) > 0$ and $Q(1) < 0$, there are roots $x_1, x_2$, and $x_3$ of $Q(x) = 0$ such that $x_1 < 0$, $0 < x_2 < 1$, and $x_3 > 1$. Therefore, the equation (19) has only one root satisfying $T > 1$. Similar to Remark 3.3 we observe that $T$ is the largest real root of the equation $Q(x^2) = 0$ and $Q(x^2)$ is positive if $x^2$ tends to infinity. Hence $Q(x^2) > 0$ for any $x > T$. Straightforward calculations show that

$$Q\left(\tan^2 \frac{\alpha_i}{2}\right) = -\tan^4 \frac{\alpha_i}{2} \cdot \left(1 + \tan^2 \frac{\alpha_{i+1}}{2}\right) \cdot \left(1 + \tan^2 \frac{\alpha_{i+2}}{2}\right) \cdot \left(1 + \tan^2 \frac{\alpha_{i+3}}{2}\right) < 0,$$

that implies $T > \tan(\alpha_i/2)$ for $i = 1, \ldots, 4$.

**Remark 4.5.** The equation (16) is equivalent to $Q(x) = 0$ under the substituting $v = (x - 1)/(x + 1)$. Hence, there is only one root $v$ of (16) satisfying $0 < v < 1$. If $T$ is the largest real root of $Q(T^2) = 0$ then $v = (T^2 - 1)/(T^2 + 1)$ and

$$v + \cos \alpha_i = \frac{T^2 - 1}{T^2 + 1} + \frac{1 - \tan^2(\alpha_i/2)}{1 + \tan^2(\alpha_i/2)} = \frac{2(T^2 - \tan^2(\alpha_i/2))}{(T^2 + 1)(1 + \tan^2(\alpha_i/2))}.$$ 

Therefore, $v + \cos \alpha_i > 0$ for $i = 1, \ldots, 4$.

### 4.3. The volume.

**Theorem 4.6.** The volume of the hyperbolic polyhedron $L(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, with $0 < \alpha_i < \pi$, $i = 1, \ldots, 4$, is given by formulae (21), (22) and (23):

$$\text{vol} L(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \frac{1}{2} \int_{v}^{1} \log \left[ \frac{1}{(1 - \nu^2)^2} \prod_{i=1}^{4} (\nu + \cos \alpha_i) \right] \frac{d\nu}{\sqrt{1 - \nu^2}}, \tag{21}$$

where $v$ satisfying $0 < v < 1$ is a root of equation (16);

$$\text{vol} L(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \int_{T}^{\infty} \log \left[ \frac{1}{t^4} \prod_{i=1}^{4} \frac{t^2 - \tan^2(\alpha_i/2)}{1 + \tan^2(\alpha_i/2)} \right] \frac{dt}{1 + t^2}, \tag{22}$$
where $T$ satisfying $T > 1$ is a root of equation (19);

\[
(23) \quad \text{vol} \mathcal{L}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \sum_{i=1}^{4} \Delta(\alpha_i/2, \theta) - 2\Delta(0, \theta) - 2\Delta(\pi/2, \theta),
\]

where $\theta$, $0 < \theta < \pi/2$, is such that $\tan \theta = T$.

**Proof.** Formulae (21), (22) and (23) give us different expressions for the volume of the same hyperbolic polyhedra depending on the choice of a principal parameter. Formula (22) corresponds to the case where the principal parameter $T$ is taken from the Tangent Rule (Proposition 4.2). The method to obtain the volume formula from the Tangent Rule is the same as in the proof of Theorem 3.5. Obviously, formula (22) arises under the same method with slight modifications. Formula (21) corresponds to the case where the principal parameter $v$ is taken from the relation between essential angles and lengths described in Proposition 4.1. Here we would like to demonstrate how to obtain the volume formula using this kind of relations between essential parameters.

Let $V = V(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \text{vol} \mathcal{L}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be the volume function. To find $V$ we will use the Schl"afli variation formula. We will need to integrate the differential form

\[
\omega = dV = -\frac{1}{2} \sum_{i=1}^{4} \ell_i d\alpha_i
\]

which is defined for $0 < \alpha_1, \alpha_2, \alpha_3, \alpha_4 < \pi$. In order to do this we will find an extended differential form $\Omega$ of five independent variables $v$ and $\alpha_i$, $i = 1, \ldots, 4$:

\[
\Omega = -\frac{1}{2} \left( \sum_{i=1}^{4} \ell_i d\alpha_i + \ell dv \right),
\]

where $\ell_i = \arccosh \frac{1 + v \cos \alpha_i}{v + \cos \alpha_i}$, $i = 1, \ldots, 4$, with $\ell = \ell(v, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$. Here $v$ plays a role of a principal parameter. We look for $\Omega$ satisfying the following properties:

1. $\Omega$ is smooth and exact in the region

\[
\mathcal{U} = \{(v, \alpha_1, \ldots, \alpha_4) : 0 < v < 1, 0 < \alpha_i < \pi, v + \cos \alpha_i > 0, i = 1, \ldots, 4\};
\]

2. $\Omega = \omega$ for all $(v, \alpha_1, \ldots, \alpha_4) \in \mathcal{U}$ satisfying equation (16).

Since $\Omega$ is supposed to be exact, fixing $i$ we have

\[
\frac{\partial \ell}{\partial \alpha_i} = \frac{\partial \ell_i}{\partial v} = \frac{\partial}{\partial v} \left( \arccosh \frac{1 + v \cos \alpha_i}{v + \cos \alpha_i} \right) = \frac{-\sin \alpha_i}{(v + \cos \alpha_i)\sqrt{1 - v^2}}.
\]

Hence

\[
\ell = \int \frac{-\sin \alpha_i}{(v + \cos \alpha_i)\sqrt{1 - v^2}} d\alpha_i = \frac{1}{\sqrt{1 - v^2}} \log(v + \cos \alpha_i) + C,
\]

where $C$ is depended only of of $v$ and $\alpha_j$, $j \neq i$. Therefore,

\[
\ell = \frac{1}{\sqrt{1 - v^2}} \log \left( \frac{1}{(1 - v^2)^2} \prod_{i=1}^{4} (v + \cos \alpha_i) \right)
\]
We note that in region \( \mathcal{U} \) equation (16) is equivalent to the following condition:

\[
\frac{1}{(1 - v^2)^2} \prod_{i=1}^{4} (v + \cos \alpha_i) = 1.
\]

If this condition is satisfied, we have \( \ell = 0 \) and consequently \( \Omega = \omega \).

By the Schlafli formula the volume function \( V \) satisfies the following system of differential equations

\[
\frac{\partial V}{\partial \alpha_i} = -\frac{\ell_i}{2}, \quad i = 1, \ldots, 4.
\]

Moreover, it was shown in [18] that if \( \alpha_2 = \alpha_4 = \pi/2 \) and \( \alpha_1 = \alpha_3 \rightarrow \pi \) then polyhedron \( \mathcal{L}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \) is collapsing to a subset of a hyperbolic plane. Hence, \( V \rightarrow 0 \) as \( \alpha_2 = \alpha_4 = \pi/2 \) and \( \alpha_1 = \alpha_3 \rightarrow \pi \).

Let us consider the function

\[
W = \frac{1}{2} \int_{v}^{1} \log \left[ \frac{1}{(1 - v^2)^2} \prod_{i=1}^{4} (v + \cos \alpha_i) \right] \frac{d\nu}{\sqrt{1 - \nu^2}}.
\]

Taking into account equations (10) and (16) by the Leibniz rule we obtain

\[
\frac{\partial W}{\partial \alpha_i} = \frac{1}{2} \int_{v}^{1} \frac{\partial}{\partial \alpha_i} \left( \log \left[ \frac{1}{(1 - v^2)^2} \prod_{i=1}^{4} (v + \cos \alpha_i) \right] \right) \frac{d\nu}{\sqrt{1 - \nu^2}}
\]

\[
= \frac{1}{2} \int_{v}^{1} \frac{-\sin \alpha_i}{v + \cos \alpha_i} \sqrt{1 - \nu^2} = \frac{1}{2} \arccosh \frac{1 + v \cos \alpha_i}{v + \cos \alpha_i} = -\frac{\ell_i}{2}.
\]

We note that if \( v \) is a root of (16) satisfying \( 0 < v < 1 \) then \( v \rightarrow 1 \) as \( \alpha_2 = \alpha_4 = \pi/2 \) and \( \alpha_1 = \alpha_3 \rightarrow \pi \). Then the condition \( W \rightarrow 0 \) follows from convergence of the integral. Thus, \( W \) satisfies the same conditions as \( V \), whence \( W = V \).

Formula (22) follows from (21) by the substitution \( \nu = (t^2 - 1)/(t^2 + 1) \), and formula (23) is a consequence of (22) in virtue of Lemma 3.4.

**Remark 4.7.** Results of numerical calculation of volumes of hyperbolic polyhedra \( \mathcal{L}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \) by Theorem 4.6 coincide with those obtained by making use of computer program \( Orb \) created by C. Hodgson and D. Heard [23].

5. The case \( n \geq 5 \)

5.1. The existence. Let us consider for \( n \geq 5 \) a polyhedron \( \mathcal{L}(\alpha_1, \alpha_2, \ldots, \alpha_n) \) with \( n \)-gonal top and \( n \)-gonal bottom and with two levels of pentagons as a lateral surface (see Fig. 6 for the projection of \( \mathcal{L}(\alpha_1, \alpha_2, \ldots, \alpha_5) \)). We will assume that essential dihedral angles equal to \( \alpha_i, 0 < \alpha_i < \pi, \quad i = 1, \ldots, n \), while all other dihedral angles equal to \( \pi/2 \).

It follows from the Rivin theorem that \( \mathcal{L}(\alpha_1, \alpha_2, \ldots, \alpha_n) \) can be realized in \( \mathbb{H}^3 \). Let us demonstrate arguments of the proof in the case \( n = 5 \). Combinatorially, the Gaussian image \( \mathcal{G}(\mathcal{L}(\alpha_1, \alpha_2, \ldots, \alpha_5)) \) is an icosahedron with five edges of lengths \( \alpha_i = \pi - \alpha_i, \quad i = 1, \ldots, 5 \), while all other edges of length \( \pi/2 \). It has twelve cone points: two for each cone angle \( 4\pi/2 + (\pi - \alpha_i) = 2\pi + (\pi - \alpha_i), \quad i = 1, \ldots, 5 \) and two with angles \( 5\pi/2 \). Therefore, at each cone point the sum of angles is greater than \( 2\pi \). Moreover, lengths of closed geodesics are greater than \( 2\pi \), because any geodesic consists of at least five segments such that length of at least four of them is equal to \( \pi/2 \). Therefore, \( \mathcal{L}(\alpha_1, \alpha_2, \ldots, \alpha_5) \) is hyperbolic polyhedron if and only if all of its essential angles are less than \( \pi/2 \).
If all $\alpha_i \to \pi$ then the polyhedron collapse to a right-angled prism in $\mathbb{H}^2 \times \mathbb{R}^1$ which has the right-angled regular $n$-gon as the bottom.

Below we will consider the symmetrical case: all essential dihedral angles are equal to $\alpha$. Denote the length of an edge corresponding to $\alpha$ by $\ell_\alpha$. For such a polyhedron with a cyclic symmetry of order $n$ we will use notation $L_n(\alpha) = L(\alpha, \alpha, \ldots, \alpha)$.

5.2. Relations between angles and lengths. Since $L_n(\alpha)$ has a cyclic symmetry of order $n$, it is enough to consider a local fragment of it. Let us denote by $\Pi_0$ and $\Sigma_0$ the bottom plane and the top plane of $L_n(\alpha)$ respectively, and by $\Pi_1, \Pi_2, \Sigma_1, \Sigma_2$ its lateral planes as in Fig. 7. Let $O$ and $O'$ be centers on the bottom and the top, respectively, which are right-angled regular $n$-gons. Let $OB$ be the line perpendicular to the side of the bottom polygon, and $OA$ be the line passing through the a vertex of the bottom polygon. Obviously, the angle between $OA$ and $OB$ is equal to $\nu = \pi/n$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6.png}
\caption{The polyhedron $L(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ and its Gaussian image.}
\end{figure}

Let $k$ and $m$ be unit vectors in $\Pi_0$ such that $k$ is normal to $OB$, $m$ is normal to $OA$, and they are oriented “outside” from the triangle $AOB$: $(k, m) = -\cos \nu$. Let $p_1, p_2, s_1, s_2$ be unit vectors normal to $\Pi_1, \Pi_2, \Sigma_1, \Sigma_2$, respectively. Since dihedral angles (if planes are intersecting) and distances (if plane are diverging) between pairs of planes $\Pi_1, \Pi_2, \Sigma_1$ and $\Sigma_2$ are known, we have

$$(p_1, p_2) = 0, \quad (s_1, p_1) = 0, \quad (s_2, p_1) = -\cosh \ell_\alpha, \quad (s_1, p_2) = -\cos \alpha, \quad (s_1, s_2) = 0.$$ 

Since $L_n(\alpha)$ has cyclic symmetry of order $n$, the plane $\Sigma_2$ is the image of the plane $\Sigma_1$ under the rotation $\rho$ about $OO'$ on the angle $2\nu = 2\pi/n$. Obviously, $\rho$ can be presented as
a composition of the reflection in the plane orthogonal to \( k \) with the reflection in the plane orthogonal to \( m \).

According to a general theory of reflections in vector spaces [1], the reflection \( r_\mu \) in the plane orthogonal to a vector \( \mu \) can be presented in the form

\[
r_\mu(x) = x - 2(x, \mu) \cdot \mu.
\]

Therefore,

\[
\rho(x) = r_m(r_k(x)) = x - 2(x, k) \cdot k - 2(x - 2(x, k) \cdot k, m) \cdot m.
\]

Since \( \Sigma_2 = \rho(\Sigma_1) \), we get:

\[
s_2 = s_1 - 2(s_1, k) \cdot k - 2(s_1 - 2(s_1, k) \cdot k, m) \cdot m = s_1 - 2(s_1, k) \cdot k - 2(s_1, m) \cdot m + 4(s_1, k)(k, m) \cdot m.
\]

Using this equality, we can represent the scalar product of vectors \( s_2 \) and \( p_1 \) as the following:

\[
(s_2, p_1) = (s_1, p_1) - 2(s_1, k)(k, p_1) - 2(s_1, m)(m, p_1) + 4(s_1, k)(k, m)(m, p_1).
\]

Since \( (s_1, p_1) = 0 \) and \( (k, p_1) = 0 \), we conclude:

\[
(s_2, p_1) = -2(s_1, m)(m, p_1) + 4(s_1, k)(k, m)(m, p_1).
\]

Recalling that \( (k, m) = -\cos \nu, (m, p_1) = -\cos(\pi/4) = -1/\sqrt{2} \), and \( (s_2, p_1) = -\cosh \ell_\alpha \), we obtain

\[
-\cosh \ell_\alpha = \sqrt{2} \cdot (s_1, m) + 2\sqrt{2} \cdot \cos \nu \cdot (s_1, k).
\]

Representing \( m = (p_2 - p_1)/\sqrt{2} \) and using \( (s_1, p_2) = -\cos \alpha \) and \( (s_1, p_1) = 0 \), we get

\[
(24) \quad \cosh \ell_\alpha = \cos \alpha - 2\sqrt{2} \cos \nu \cdot (s_1, k).
\]

To find \( (s_1, k) \) let us take the scalar product of vectors \( s_2 \) and \( s_1 \), where \( s_2 \) is presented as the linear combinations of \( s_1, k \), and \( m \), we have

\[
(s_2, s_1) = (s_1, s_1) - 2(s_1, k)(k, s_1) - 2(s_1, m)(m, s_1) + 4(s_1, k)(k, m)(m, s_1).
\]

Since \( (s_2, s_1) = 0 \), \( (s_1, s_1) = 1 \), \( (k, m) = -\cos \nu \), and \( (s_1, m) = -\frac{1}{\sqrt{2}} \cos \alpha \), we obtain

\[
2(s_1, k)^2 - 2\sqrt{2} \cdot \cos \alpha \cdot \cos \nu \cdot (s_1, k) + \cos^2 \alpha - 1 = 0.
\]

Solving this quadratic equation we have

\[
(s_1, k) = \frac{1}{\sqrt{2}} \left( \cos \alpha \cos \nu \pm \sqrt{\cos^2 \alpha \cos^2 \nu + \sin^2 \alpha} \right),
\]

and by (24):

\[
\cosh \ell_\alpha = \cos \alpha - 2 \cos \alpha \cos^2 \nu \mp 2 \cos \nu \sqrt{\cos^2 \alpha \cos^2 \nu + \sin^2 \alpha} = -\cos \alpha \cos(2\nu) \mp 2 \cos \nu \sqrt{\cos^2 \alpha \cos^2 \nu + \sin^2 \alpha}
\]

To guarantee \( \cosh \ell_\alpha > 1 \) we have to choose “+” for the square root. Thus, we obtained

**Proposition 5.1.** The essential angles and lengths of the hyperbolic polyhedron \( \mathcal{L}_n(\alpha) \), \( n \geq 5 \), with \( 0 < \alpha < \pi \), satisfy the following relation:

\[
(25) \quad \cosh \ell_\alpha = -\cos \alpha \cos(2\pi/n) + 2 \cos(\pi/n) \sqrt{\cos^2 \alpha \cos^2(\pi/n) + \sin^2 \alpha}.
\]
In particular, if $\alpha = \pi/2$ then $\cosh \ell_\alpha = 2 \cos(\pi/n)$. The formula (25) also holds for $n = 4$ with $0 < \alpha < \pi$ and for $n = 3$ with $0 < \alpha < \pi/2$.

5.3. **The volume formula.** Substituting the expression for the length $\ell_\alpha$ into the Schlafli formula, we will obtain the expression for the volume of the polyhedron.

**Theorem 5.2.** The volume $\text{vol} \mathcal{L}_n(\alpha)$ of the hyperbolic polyhedron $\mathcal{L}_n(\alpha)$, $n \geq 5$, where $0 < \alpha < \pi$, is given by the following formula:

$$\frac{n}{2} \int_\alpha^\pi \arccosh \left[ -\cos \mu \cos(2\pi/n) + 2 \cos(\pi/n) \sqrt{\cos^2 \mu \cos^2(\pi/n) + \sin^2 \mu} \right] d\mu.$$ 

Recall that the polyhedron $\mathcal{L}_n(\pi/2)$ is right-angled and is the same as the polyhedron $R(n)$ discussed in the introduction. Polyhedra $\mathcal{L}_5(\pi/2) = R(5)$ and $\mathcal{L}_6(\pi/2) = R(6)$ are presented in Fig. 2. The volume formula for $\mathcal{L}_n(\pi/2)$ was obtained in [34] in the context of volumes of Lobell manifolds.

**Proposition 5.3.** [34] The volume of the polyhedron $\mathcal{L}_n(\pi/2)$, $n \geq 5$, is given by

$$\text{vol} \mathcal{L}_n(\pi/2) = \frac{n}{2} \left( 2 \Lambda(\theta) + \Lambda(\theta + \pi/n) + \Lambda(\theta - \pi/n) - \Lambda(2\theta - \pi/2) \right),$$

where $\theta = \pi/2 - \arccos \frac{1}{2 \cos(\pi/n)}$.

In particular, if $L$ is the Lobell manifold obtained in [12] from eight copies of $R(6)$ then

$$\text{vol} L = 24 \left( 2 \Lambda(\theta) + \Lambda(\theta + \pi/6) + \Lambda(\theta - \pi/6) - \Lambda(2\theta - \pi/2) \right),$$

where $\theta = \pi/2 - \arccos \frac{1}{\sqrt{3}}$. Hence, $\text{vol} L = 48.184368\ldots$.

Also, it is easy to see the following asymptotic formula:

$$\text{vol} \mathcal{L}_n(\pi/2) \sim \frac{5}{4} \cdot v_{max}^3 \cdot n \quad n \to \infty,$$

where $v_{max}^3 = 2 \Lambda(\pi/6) = 1.014\ldots$ is the maximal volume of a hyperbolic 3-simplex, that is, the volume of a regular ideal tetrahedron in $\mathbb{H}^3$.

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