Packings by translation balls in $\widetilde{\text{SL}_2(\mathbb{R})}$

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Abstract. For one of Thurston model spaces, $\widetilde{\text{SL}_2(\mathbb{R})}$, we discuss translation balls and packing that space by such balls in contrast to the packing by standard (geodesic) balls. We present an infinite family of packings generated by discrete groups of isometries, and observe numerical results on their densities. In particular, we found packings whose densities are close to the upper bound density for ball packings in the hyperbolic 3-space.


Keywords. Thurston geometries, $\widetilde{\text{SL}_2(\mathbb{R})}$ geometry, ball packings, tiling, regular prism, volume in $\widetilde{\text{SL}_2(\mathbb{R})}$ space.

1. Introduction and preliminaries

1.1. Ball packings in spaces of constant curvature

In the present paper we consider ball (sphere) packings in the non-Euclidean 3-dimensional space $\widetilde{\text{SL}_2(\mathbb{R})}$. Since the subject of sphere packing has a long story, let we recall some known results. The story of studying density of sphere packing in the 3-dimensional Euclidean space $\mathbb{E}^3$ arises from the famous Johannes Kepler’s monograph “The Six-Cornered Snowflake” (1611), where he conjectured: No packing of spheres of the same radius has a density greater than the face-centered cubic packing. This is the oldest problem in discrete geometry and is an important part of the Hilbert’s 18-th problem. In 1953, László Fejes Tóth reduced the Kepler conjecture to an enormous calculation procedure that involved specific cases, and later he suggested that computers might be helpful for solving the problem. In this

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way the Kepler conjecture has finally been proved by Thomas Hales [6]. Thus, the
greatest packing density for $E^3$ is equal to $\pi/\sqrt{18} = 0.74048$. Here and below we
present only five digits after the point for constant under discussion. Let $X$ be one
of the $n$-dimensional spaces of constant curvature: $E^n$, $H^n$, or $S^n$ ($n \geq 2$). Let $d_n(r)$
be the density of $n+1$ spheres of radius $r$ mutually touching one another with
respect to the simplex spanned by the centres of the spheres. L. Fejes Tóth and
H. S. M. Coxeter conjectured that in an $n$-dimensional space of constant curvature
the density of packing balls of radius $r$ can not exceed $d_n(r)$. The 2-dimensional
spherical case was done by L. Fejes Tóth [5]. In the case of the Euclidean space $E^n$
the conjecture has been proved by C. A. Rogers [16]. In [3] K. Böröczky proved
the following result.

Theorem 1.1. [3] In an $n$-dimensional space of constant curvature consider a pack-
ing of spheres of radius $r$. In spherical space suppose that $r < \pi/4$. Then the
density of each sphere in its Dirichlet – Voronoi cell cannot exceed the density of
$n+1$ spheres of radius $r$ mutually touching one another with respect to the simplex
spanned by their centres.

It was shown in [2] that in $H^3$ the upper bound density is equal to 0.85326 that can
be realized by the horoball packing corresponding to tiling of $H^3$ by ideal regular
tetrahedra. However, this construction of horoballs, under the Coxeter reflectional
symmetry group (3, 3, 6) in $H^3$ does not lead to a ball packing whose density is close
as possible to the above upper bound. Moreover, we do not know about any $H^3$
packing with equal balls, either group generated packing (see the next section) or
other one, whose density would be close to this upper bound. Furthermore, we do
not know any similar general upper bound in the other Thurston geometries (see
the next subsection 1.2), although the second author has analogous conjectures on
the upper bound in Nil and $S^2 \times \mathbb{R}$, respectively (see subsection 1.3).

1.2. Group generated packing

It is known due to Thurston that there are exactly eight 3-dimensional so-called
model geometries [17]. We already mentioned three of them: $E^3$, $S^3$, and $H^3$.
Another five are: $S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$, $\text{SL}_2(\mathbb{R})$, Nil, and Sol. The uniform interpretation of
all eight Thurston geometries in the projective 3-sphere and 3-space was done by
E. Molnár in [9]. Basing on this interpretation some results on sphere packing of
Thurston model spaces were obtained recently. Let $X$ be one of Thurston model
spaces, and let $d(P,Q)$ be the standard (i. e. geodesic, or later, also translation)
distance between points $P, Q \in X$. A set $\mathcal{B} \subset X$ of balls is said to be a ball packing
if any two balls $B_1, B_2 \in \mathcal{B}$ are either touching or disjoint. We recall that a group
$\Gamma$ acts on a space $X$ properly discontinuously if for any compact $C \subset X$ the set
$\{g \in \Gamma : g(C) \cap C \neq \emptyset\}$ is finite. For brevity we shall say that a group $\Gamma$ is discrete
instead of saying that $\Gamma$ acts properly discontinuously. For a discrete group $\Gamma$ of
isometries of $X$ and a point $P \in X$ denote by $\mathcal{D}(P)$ a Dirichlet — Voronoi cell of
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\( \Gamma \) centred in \( P \):

\[
\mathcal{D}(P) = \{ Q \in \mathbb{X} : d(P, Q) \leq d(g(P), Q) \text{ for all } g \in \Gamma \}.
\]

In the following we suppose for \( \Gamma \) and \( P \) that \( \mathcal{D}(P) \) has a finite volume. Let \( B_P \) be a ball in \( \mathcal{D}(P) \) centred in \( P \). Then the orbit set \( \Gamma(B_P) = \{ g(B_P) : g \in \Gamma \} \) is a ball packing in \( \mathbb{X} \). Such packing \( \mathcal{B} = \Gamma(B_P) \) will be referred as a group \( \Gamma \) generated packing.

**Definition 1.2.** The **density** of a group \( \Gamma \) generated packing \( \Gamma(B_P) \) is the following ratio:

\[
\delta = \frac{\text{vol}(B_P)}{\text{vol}(\mathcal{D}(P))}, \quad (1.1)
\]

where \( \mathcal{D}(P) \) is the Dirihlet – Voronoi cell for \( \Gamma \) centred in \( P \) and \( B_P \subset \mathcal{D}(P) \).

Suppose that \( \Gamma \) and \( P \in \mathbb{X} \) are fixed. We say that a group \( \Gamma \) generated packing \( \Gamma(B_P) \) is **dense** if its density is maximal. Denote by \( \Gamma_P \) the stabilizer of a point \( P \in \mathbb{X} \) in the group \( \Gamma \). Then the radius \( \rho \) of any ball \( B_P \subset \mathcal{D}(P) \) satisfies the following inequality:

\[
\rho(P) \leq \frac{1}{2} \min_{g \in \Gamma \setminus \Gamma_P} d(P, g(P)).
\]

Obviously, if \( \rho(P) \) realises the equality, then the corresponding group \( \Gamma \) generated packing is dense. Then we can look for the **maximal (optimal) density** depending on \( P \) and on the possibly free parameters for \( \Gamma \), which can be a difficult problem.

### 1.3. Ball packings in Thurston model spaces

Group generated packings in Thurston model spaces different of constant curvature spaces have been intensively investigated last years. The lattice-like sphere packing of the \( \text{Nil} \) space of the best known density 0.78085 was found by J. Szirmai in [18]. It seems to be surprising that this density is greater than 0.74048 corresponding to the Euclidean case. Two-parameter geodesic ball packings in \( \text{SL}_2(\mathbb{R}) \) corresponding to the later groups \( pq2_1 \) of prismatic tillings were constructed by E. Molnár and J. Szirmai in [12]. Their record density was 0.567362 for \( (p, q) = (8, 10) \).

In the space \( \mathbb{H}^2 \times \mathbb{R} \) the sphere packing corresponding to certain generalized Coxeter group with density 0.60726 was found by J. Szirmai in [20]. In the space \( \mathbb{S}^2 \times \mathbb{R} \) the sphere packing corresponding to a generalized Coxeter group with density 0.87757 was found by J. Szirmai in [23]. Let us recall the following conjecture posed in [23].

**Conjecture 1.3.** Let \( \mathcal{B} \) be an arbitrary packing with congruent balls in a Thurston model space \( \mathbb{X} \), where \( \mathcal{B} \) is generated by a discrete group of isometries of \( \mathbb{X} \). Then its density doesn’t exceed 0.87757 realized in \( \mathbb{S}^2 \times \mathbb{R} \).
1.4. The double link group $G_{p,q}$ and its orbifold in $\widetilde{\text{SL}_2}\mathbb{(R)}$

Denote by $G_{p,q}$, for integer $p, q \geq 3$, a two-generated group with the following presentation:

$$G_{p,q} = \langle a, b : a^p = b^q = ababa^{-1}b^{-1}a^{-1}b^{-1} = 1 \rangle. \quad (1.2)$$

We recall that the group $\langle a, b : ababa^{-1}b^{-1}a^{-1}b^{-1} = 1 \rangle$ is a fundamental group of the 2-component link $4_1^2$ (in standard knot theory notations), presented in Fig. 1. This link is also known as the double-link. Generators $a$ and $b$ correspond to meridian loops around link components. So, $G_{p,q}$ is orbifold group of the 3-orbifold $O_{p,q}$ with underlying space the 3-sphere, the singular set $4_1^2$, and singular angles $2\pi/p$ and $2\pi/q$ around its components. Topological properties of cyclic branched coverings of orbifolds $O_{p,p}$ were investigated in [14, 15].

In [13] we have already demonstrated that for $p, q \geq 3$, satisfying $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$, the realization of the group $G_{p,q}$ as a group of isometries of $\widetilde{\text{SL}_2}\mathbb{(R)}$ (using the projective model of that space). Moreover, we described the realization of its fundamental polyhedron. Thus, the orbifold $O_{p,q}$ admits $\widetilde{\text{SL}_2}\mathbb{(R)}$–geometry for those parameters. (See [1] about geometrization of 3-orbifolds.)

In [22] that group $G_{p,q}$, denoted by $pq2_1$, appeared in a context of prismatic tilings as the space group of $\widetilde{\text{SL}_2}\mathbb{(R)}$ generated by a $p$–gonal rotation $a$ and a $q$–gonal rotation $b$ such that their product $h = ab$ is a half-screw $2_1$. This prismatic tiling was used in [12] to construct an infinite family of group $G_{p,q}$ generated ball packings in $\widetilde{\text{SL}_2}\mathbb{(R)}$.

As well as standard (geodesic) balls in Thurston model spaces, E. Molnár initiated (see, for example [7]) to consider translation balls which correspond to calculating distances along translation curves instead of geodesic curves in the standard case. Thus, one can talk about translation distance, translation curves, and translation balls. For spaces $\mathbb{E}^3$, $\mathbb{S}^3$, $\mathbb{H}^3$, $\mathbb{S}^2 \times \mathbb{R}$, and $\mathbb{H}^2 \times \mathbb{R}$ the translation objects are the same as the standard ones, but for the other three Thurston model spaces they are different.
In the present paper we discuss packing by equal translation balls in $\tilde{\text{SL}}_2(\mathbb{R})$. We define the translations of $\tilde{\text{SL}}_2(\mathbb{R})$ as the specific isometry group acting transitively on points; its invariant arc–length–square; and further the translation curve, the translation distance, spheres and balls. Then we construct the infinite family of packing in $\tilde{\text{SL}}_2(\mathbb{R})$ by translation balls generated by the group $G_{p,q} = pq\mathbf{2}_1$ action. Of cause, any translation ball is a subset of a standard ball of the same centre and radius. But this fact doesn’t give us deciding information about densities, because of the difference of geodesic and translation distances, and so, shape difference of corresponding Dirihlet – Voronoi cells, depending on parameters $p$ and $q$. Indeed, so far we got larger densities for translation ball packings for the groups $pq\mathbf{2}_1$ above (see our Table 1 for results of computations, where the best obtained case is indicated by bold). But nevertheless, it seems an open question if there is a packing in $\tilde{\text{SL}}_2(\mathbb{R})$ by translation balls whose density is close to above mentioned constant $0.85326$ (which is the upper bound density in $\mathbb{H}^3$).

2. Translations in $\tilde{\text{SL}}_2(\mathbb{R})$

2.1. The hyperboloid model in the projective 3–sphere

Since $\text{SL}_2\mathbb{R}$ is a 3-dimensional Lie group, its universal covering space $\tilde{\text{SL}}_2(\mathbb{R})$ is also a Lie group and admits a Riemann metric invariant under (say) right multiplication.

We recall that the geometry of $\tilde{\text{SL}}_2(\mathbb{R})$ arises naturally as geometry of a fibre line bundle over a base hyperbolic plane $\mathbb{H}^2$. Consider $\tilde{C} = \mathbb{C} \cup \{\infty\}$ as the complex projective line with the correspondence $z \sim (1, z) \sim (z^0, z^1)$. Using the homogeneous coordinates the right action $(w^0, w^1) = (z^0, z^1)g$ by $g \in \text{SL}_2(\mathbb{R})$ can be written as

$$(w^0, w^1) = (z^0, z^1) \begin{pmatrix} \delta & \beta \\ \gamma & \alpha \end{pmatrix} = (z^0 \delta + z^1 \gamma, z^0 \beta + z^1 \alpha), \quad \text{for} \quad g = \begin{pmatrix} \delta & \beta \\ \gamma & \alpha \end{pmatrix}.$$ 

This action corresponds to the standard linear-fractional action $z \mapsto \frac{\beta + z\alpha}{\delta + z\gamma}$, where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $\alpha\delta - \beta\gamma = 1$. If two elements of $\text{SL}_2(\mathbb{R})$ have equal quotients in $\text{PSL}_2(\mathbb{R})$, then they represent the same action.

Let $\mathcal{P}^3$ be the projective space and let $\mathcal{PS}^3$ be the projective sphere with coordinates $(x^0; x^1; x^2; x^3)$. We correspond to each element $\begin{pmatrix} \delta & \beta \\ \gamma & \alpha \end{pmatrix} \in \text{PSL}_2(\mathbb{R})$ projective coordinates according to the following formulae:

$$x^0 = \frac{\alpha + \delta}{2}, \quad x^1 = \frac{\beta - \gamma}{2}, \quad x^2 = \frac{\beta + \gamma}{2}, \quad x^3 = \frac{\alpha - \delta}{2}.$$
Considering the inverse, we have
\[ \alpha = x^0 + x^3, \quad \beta = x^1 + x^2, \quad \gamma = -x^1 + x^2, \quad \delta = -x^3 + x^0 \]
up to a (multiplicative) projective freedom. Since \( \beta \gamma - \alpha \delta < 0 \), we get
\[ -(x^0)^2 - (x^1)^2 + (x^2)^2 + (x^3)^2 < 0. \quad (2.1) \]
For \( X(x^0; x^1; x^2; x^3) \) and \( Y(y^0; y^1; y^2; y^3) \) define the scalar product as the follows:
\[ (X, Y) = -x^0 y^0 - x^1 y^1 + x^2 y^2 + x^3 y^3. \quad (2.2) \]
The inequality (2.1) defines the interior of the one-sheeted hyperboloid solid \( \mathcal{H} \) in the
projective sphere:
\[ \mathcal{H} = \{ X \in \mathcal{P}S^3 : (X, X) < 0 \}. \quad (2.3) \]
In (inhomogeneous) Euclidean coordinates \( x = x^1/x^0, \ y = x^2/x^0, \ z = x^3/x^0 \) the
inequality (2.1) looks as the following:
\[-x^2 + y^2 + z^2 < 1. \quad (2.4) \]
Let \( \{ e_i, \ i = 0, 1, 2, 3 \} \) be the basis in \( \mathbb{R}^4 \) that can be considered as a simplex
coordinate system in \( \mathcal{P}S^3 \). Vertices of the corresponding coordinate simplex are the origin
\( E_0 = (1; 0; 0; 0) \) and three ideal points of the axes: \( E_\infty^0(0; 1; 0; 0), \ E_\infty^1(0; 0; 1; 0), \) and \( E_\infty^2(0; 0; 0; 1) \). For \( \mathcal{P}^3 \), as well as for \( \mathcal{P}S^3 \), consider the one-parameter screw
collineation group \( \mathcal{S} \) consisting of elements \( S_\varphi \) of the form
\[ S_\varphi : \begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \mapsto \begin{pmatrix} \cos \varphi & \sin \varphi & 0 & 0 \\ -\sin \varphi & \cos \varphi & 0 & 0 \\ 0 & 0 & \cos \varphi & -\sin \varphi \\ 0 & 0 & \sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \quad (2.5) \]
with \( -\pi/2 < \varphi \leq \pi/2 \) (mod \( \pi \)) in the case of \( \mathcal{P}^3 \), and with \( -\pi < \varphi \leq \pi \) (mod \( 2\pi \)) in the case of \( \mathcal{P}S^3 \). This group action preserves the signature \((-,-,+,+)\) of the scalar product in (2.2) and so, it preserves the hyperboloid solid \( \mathcal{H} \subset \mathcal{P}S^3 \). For any two different points their \( \mathcal{S} \)-orbits are either mutually skew lines or they coincide (see Fig. 2 and [9] for details).

The plane \( H_\infty^2 := E_0 E_\infty^0 E_\infty^1 \) (that corresponds to the condition \( x^1 = 0 \)) intersects \( \mathcal{H} \) and represents the hyperbolic plane \( \mathbb{H}^2 \) with the absolute
\[ \{(x^0; x^1; x^2; x^3) : -x^0 x^0 + x^2 x^2 + x^3 x^3 = 0, \quad x^1 = 0 \}. \]
For any point \( Y(y^0; y^1; y^2; y^3) \in \mathcal{H} \) its orbit \( \mathcal{S}(Y) \) under the group action is the following set depending of the parameter \( \varphi \):
\[ \mathcal{S}(Y) = \{(y^0 \cos \varphi - y^1 \sin \varphi; y^0 \sin \varphi + y^1 \cos \varphi; y^2 \cos \varphi + y^3 \sin \varphi; -y^2 \sin \varphi + y^3 \cos \varphi) \}. \quad (2.6) \]
The set \( \{ S(Y) : Y \in \mathcal{H} \} \) forms a fiber line bundle in \( \mathcal{H} \), and the plane \( H_0^2 \) is its base. Indeed, the intersection point \( Z(z_0; z_1; z_2; z_3) \) of the plane \( H_0^2 \) with the line \( S(Y) \) passing through \( Y(y_0; y_1; y_2; y_3) \) has the following coordinates:

\[
\begin{align*}
    z_0 &= y^0 y_0 + y^1 y_1; \\
    z_1 &= 0; \\
    z_2 &= y^0 y_2 - y^1 y_3; \\
    z_3 &= y^0 y_3 + y^1 y_2.
\end{align*}
\]

In this case we say that the bundle line \( S(Y) \) grows up at a point \( Z \). Let \( \widetilde{\mathcal{H}} \) be the universal covering space of the hyperboloid \( \mathcal{H} \) constructed by extending the fixed point free action of \( S_\varphi \) for any \( \varphi \in \mathbb{R} \). Then \( \widetilde{\mathcal{H}} \) is modelling our space:

\[
\widetilde{\mathcal{H}} = \widetilde{\text{SL}_2(\mathbb{R})} = \widetilde{T_1H^2}.
\]

Thus, every bundle line that grows up at a point of the hyperbolic base plane \( H_0^2 \) presents the revolving tangent vector from \( \widetilde{T_1H^2} \) (see [9]).

### 2.2. Isometries of \( \widetilde{\text{SL}_2(\mathbb{R})} \)

The isometry group \( G \) of \( \widetilde{\mathcal{H}} \) is defined by collineations, acting on the right on points as row matrices \((x^0; x^1; x^2; x^3)\) and leaving the polarity (by the scalar product (2.2)) and the line bundle invariant. In the basis \( \{ e_i, i = 0, 1, 2, 3 \} \) any element of \( G \) can be presented by a matrix

\[
A = \begin{pmatrix} a_0^0 & a_0^1 & a_0^2 & a_0^3 \\ a_1^0 & a_1^1 & a_1^2 & a_1^3 \\ a_2^0 & a_2^1 & a_2^2 & a_2^3 \\ a_3^0 & a_3^1 & a_3^2 & a_3^3 \end{pmatrix}
\]

with \( \det(A) = 1 \) and the following conditions:

\[
\begin{align*}
    a_0^0 &= \pm a_1^1, & a_0^2 &= \pm a_1^3, & a_0^3 &= \pm a_1^0; \\
    a_1^0 &= \mp a_0^1, & a_1^2 &= \mp a_0^3, & a_1^3 &= \pm a_0^2; \\
    a_2^0 &= \mp a_0^2, & a_2^1 &= \mp a_0^3, & a_2^3 &= \pm a_0^1; \\
    a_3^0 &= \mp a_0^3, & a_3^1 &= \mp a_0^2, & a_3^2 &= \pm a_0^1; \\
\end{align*}
\]
and
\[-(a_0^0)^2 - (a_0^1)^2 + (a_0^2)^2 + (a_0^3)^2 = -1;\]
\[-(a_1^0)^2 - (a_1^1)^2 + (a_1^2)^2 + (a_1^3)^2 = 1;\]
\[-a_0^0 a_0^1 + a_0^2 a_0^2 + a_0^3 a_0^3 = 0;\]
\[-a_0^0 a_2^2 + a_0^2 a_2^2 + a_0^3 a_2^2 = 0.\] (2.9)

Of course, we allow the projective freedom by positive proportionality. Let us denote by \(G^+\) the subgroup of \(G\) corresponding to the choice of the upper sign in formulae (2.8). For any \(B \in S\) if \(A^{-1}BA = B\) then \(A \in G^+\), and if \(A^{-1}BA = B^{-1}\) then \(A \in G \setminus G^+\).

### 2.3. Translation curves as straight lines

We recall some basic facts about translation curves in \(\widetilde{SL_2(\mathbb{R})}\) following [7]. For any point \(X(x^0; x^1; x^2; x^3) \in \mathcal{H}\) (and later also for points in \(\tilde{\mathcal{H}}\)) the translation map from the origin \(E_0(1; 0; 0; 0)\) to \(X\) is defined by the translation matrix \(T\),

\[
T = \begin{pmatrix}
    x^0 & x^1 & x^2 & x^3 \\
    -x^1 & x^0 & x^3 & -x^2 \\
    x^2 & x^3 & x^0 & x^1 \\
    x^3 & -x^2 & -x^1 & x^0
\end{pmatrix}. \tag{2.10}
\]

Its inverse is

\[
T^{-1} = \frac{1}{\det T} \begin{pmatrix}
    x^0 & -x^1 & -x^2 & -x^3 \\
    x^1 & x^0 & -x^3 & x^2 \\
    -x^2 & -x^3 & x^0 & -x^1 \\
    -x^3 & x^2 & x^1 & x^0
\end{pmatrix}. \tag{2.11}
\]

It is easy to guarantee by

\[-x^0 x^0 - x^1 x^1 + x^2 x^2 + x^3 x^3 = -1\]

that the matrix \(T\) satisfies (2.8) and (2.9) and \(T \in G^+\).

Let us consider for a given vector \((q; u; v; w)\) a curve \(C(t) = (x^0(t); x^1(t); x^2(t); x^3(t))\), \(t \geq 0\), in \(\mathcal{H}\) starting at the origin: \(C(0) = E_0(1; 0; 0; 0)\) and such that

\[
\dot{C}(0) = (\dot{x}^0(0); \dot{x}^1(0); \dot{x}^2(0); \dot{x}^3(0)) = (q; u; v; w),
\]

where \(\dot{C}(t) = (\dot{x}^0(t); \dot{x}^1(t); \dot{x}^2(t); \dot{x}^3(t))\) is the tangent vector at any point of the curve. For \(t \geq 0\) there exists a matrix

\[
T(t) = \begin{pmatrix}
    x^0(t) & x^1(t) & x^2(t) & x^3(t) \\
    -x^1(t) & x^0(t) & x^3(t) & -x^2(t) \\
    x^2(t) & x^3(t) & x^0(t) & x^1(t) \\
    x^3(t) & -x^2(t) & -x^1(t) & x^0(t)
\end{pmatrix}. \tag{2.12}
\]

which defines the translation from \(C(0)\) to \(C(t)\):

\[
C(0) \cdot T(t) = C(t), \quad t \geq 0. \tag{2.13}
\]

The \(t\)-parametrized family \(T(t)\) of translations is used in the following definition.
Definition 2.1. The curve $C(t)$, $t \geq 0$, is said to be a translation curve if
\[ \dot{C}(0) \cdot T(t) = \dot{C}(t), \quad t \geq 0. \] (2.14)

In coordinates this relation can be written as follows:
\[
(q, u, v, w) \cdot \begin{pmatrix}
    x^0(t) & x^1(t) & x^2(t) & x^3(t) \\
    -x^1(t) & x^0(t) & -x^2(t) & x^3(t) \\
    x^2(t) & x^1(t) & x^0(t) & x^1(t) \\
    -x^3(t) & -x^2(t) & -x^1(t) & x^0(t)
\end{pmatrix} = (\dot{x}^0(t), \dot{x}^1(t), \dot{x}^2(t), \dot{x}^3(t)).
\] (2.15)

By rearranging the left part of the equation we get
\[
(x^0(t), x^1(t), x^2(t), x^3(t)) \cdot \begin{pmatrix}
    q & u & v & w \\
    -u & q & -w & v \\
    v & -w & q & -u \\
    w & v & u & q
\end{pmatrix} = (\dot{x}^0(t), \dot{x}^1(t), \dot{x}^2(t), \dot{x}^3(t)).
\] (2.16)

Denoting by
\[
Q = \begin{pmatrix}
    q & u & v & w \\
    -u & q & -w & v \\
    v & -w & q & -u \\
    w & v & u & q
\end{pmatrix},
\] (2.17)

we can rewrite this equation briefly in the form
\[ C(t) \cdot Q = \dot{C}(t), \quad t \geq 0. \] (2.18)

This equation is said to be the translation dynamical system.

We remark that the translation dynamical system (2.16) is considered up to an exponential function. In particular, if $x^0(t) \neq 0$ then
\[ (x^0(t); x^1(t); x^2(t); x^3(t)) \sim (1; x(t); y(t); z(t)), \]
where $x(t) = x^1(t)/x^0(t)$, $y(t) = x^2(t)/x^0(t)$, $z(t) = x^3(t)/x^0(t)$. The triple of functions $(x(t), y(t), z(t))$ represents the Euclidean coordinates.

From the characteristic equation $\det(Q - \lambda I) = 0$, having the form
\[ [(q - \lambda)^2 - (-u^2 + v^2 + w^2)]^2 = 0, \]
one can find solutions of (2.16), depending on $(q, u, v, w)$. It was done in [7], where solutions split into the following three cases.

Case I: $-u^2 + v^2 + w^2 > 0$. Let $0 < a \in \mathbb{R}$ be such that $a^2 = -u^2 + v^2 + w^2$. Then any solution of (2.16) can be presented as
\[
(x^0(t); x^1(t); x^2(t); x^3(t)) = e^{qt} \left( \cosh(at); \frac{u}{a} \sinh(at); \frac{v}{a} \sinh(at); \frac{w}{a} \sinh(at) \right),
\] (2.19)
hence in Euclidean coordinates:
\[ x(t) = \frac{u}{a} \tanh(at), \quad y(t) = \frac{v}{a} \tanh(at), \quad z(t) = \frac{w}{a} \tanh(at). \] (2.20)
This solution can be extended for all \( t \in \mathbb{R} \) along a straight line in \( \tilde{\text{SL}}_2(\mathbb{R}) \) (a segment of the hyperboloid solid \( \mathcal{H} \)). This is the case of a \( \mathbb{H}^2 \)-direction curve.

**Case II:** \(-u^2 + v^2 + w^2 < 0\). Let \( 0 < a \in \mathbb{R} \) be such that \(-a^2 = -u^2 + v^2 + w^2\). Then any solution of (2.16) can be presented as

\[
(x^0(t); x^1(t); x^2(t); x^3(t)) = e^{at} \left( \cos(at) + \frac{u}{a} \sin(at); \frac{v}{a} \sin(at; \frac{w}{a} \sin(at) \right),
\]

hence in Euclidean coordinates:

\[
x(t) = \frac{u}{a} \tan(at), \quad y(t) = \frac{v}{a} \tan(at), \quad z(t) = \frac{w}{a} \tan(at).
\]

This solution can be extended for all \( t \in \mathbb{R} \) along a straight line in \( \tilde{\text{SL}}_2(\mathbb{R}) \) (in the sense of the universal cover \( \tilde{\mathcal{H}} \) of the hyperboloid \( \mathcal{H} \) in \( \mathcal{PS}^3 \)). This is a fibre-direction curve.

**Case III:** \(-u^2 + v^2 + w^2 = 0\). This case corresponds to an asymptotic direction. The unique solution of (2.16) is a straight line on the asymptotic cone of \( \mathcal{H} \), passing through the origin \( E_0 \) and ideal points \((0; u; v; w)\) and \((0; -u; -v; -w)\) in the sense of \( \mathcal{PS}^3 \). In the Euclidean coordinates we have

\[
x(t) = ut, \quad y(t) = vt, \quad z(t) = wt.
\]

As we see, the translation curves are straight lines in the projective embedding of \( \tilde{\text{SL}}_2(\mathbb{R}) \). Summarising the above cases the following result holds.

**Proposition 2.2.** [7] Translation curves in the one-parted hyperboloid model of \( \tilde{\text{SL}}_2(\mathbb{R}) \)-geometry are characterized by formulae (2.20), (2.22), and (2.23).

### 2.4. Translation distance

It was observed above that for any \( X(x^0; x^1; x^2; x^3) \in \tilde{\mathcal{H}} \) there is a suitable transformation \( T^{-1} \), given by (2.11) which sent \( X \) to the origin \( E_0 \) along a translation curve.

**Definition 2.3.** A translation distance \( \rho(E_0, X) \) between the origin \( E_0(1; 0; 0; 0) \) and the point \( X(1; x; y; z) \) is the length of a translation curve connecting them.

For a given translation curve \( C = C(t) \) the initial unit tangent vector \((u, v, w)\) (in Euclidean coordinates) at \( E_0 \) can be presented as

\[
u = \sin \alpha, \quad v = \cos \alpha \cos \lambda, \quad w = \cos \alpha \sin \lambda,
\]

for some \(-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}\) and \(-\pi < \lambda \leq \pi\). In \( \tilde{\mathcal{H}} \) this vector is of length square \(-u^2 + v^2 + w^2 = \cos 2\alpha\). We always can assume that \( C \) is parametrized by the translation arc-length parameter \( t = s \geq 0 \). Then coordinates of a point \( X(x; y; z) \) of \( C \), such that the translation distance between \( E_0 \) and \( X \) equals \( s \), depend on \((\lambda, \alpha, s)\) as geographic coordinates according to the above considered three cases as follows.
Case I: \(0 < -u^2 + v^2 + w^2\) (the case of a \(\mathbb{H}^2\)-direction). We have \(s(E_0, X) \geq 0\) and
\[
(x(s); y(s); z(s)) = \frac{\tanh(s \sqrt{2\alpha})}{\sqrt{\cos 2\alpha}} (\sin \alpha, \cos \alpha \cos \lambda, \cos \alpha \sin \lambda) \quad (2.25)
\]
where \(-u^2 + v^2 + w^2 = -\sin^2 \alpha + \cos^2 \alpha = \cos 2\alpha\) for \(-\frac{\pi}{4} < \alpha < \frac{\pi}{4}\).

Case II: \(-u^2 + v^2 + w^2 < 0\) (the case of a fibre direction). We have \(s(E_0, X) > 0\) and
\[
(x(s); y(s); z(s)) = \frac{\tan(s \sqrt{-\cos 2\alpha})}{\sqrt{-\cos 2\alpha}} (\sin \alpha, \cos \alpha \cos \lambda, \cos \alpha \sin \lambda) \quad (2.26)
\]
where \(u^2 - v^2 - w^2 = \sin^2 \alpha - \cos^2 \alpha = -\cos 2\alpha\) for \(-\frac{\pi}{4} < \alpha < \frac{\pi}{4}\) or \(\frac{\pi}{4} < \alpha < \frac{\pi}{2}\).

Firstly, \(0 \leq s < \frac{\pi}{4}\) holds, then \(s\) will be extended to \(\mathbb{R}_+\) by the universal cover \(\tilde{\mathcal{H}} \sim \text{SL}_2(\mathbb{R})\).

Case III: \(-u^2 + v^2 + w^2 = 0\) (the case of an asymptotic direction). We have \(s(E_0, X) \geq 0\) and
\[
(x(s); y(s); z(s)) = \frac{\sqrt{2}s}{2} (\pm 1, \cos \lambda, \sin \lambda) \quad (2.27)
\]

since \(\alpha = \pm \frac{\pi}{4}, u = \pm \frac{\sqrt{2}}{2}, v = \frac{\sqrt{2}}{2} \cos \lambda, \) and \(w = \frac{\sqrt{2}}{2} \sin \lambda\).

Fixing \(s = \rho\) as a radius and considering coordinates \(\lambda\) and \(\alpha\) from (2.24) as a longitude and an altitude, we use (2.25), (2.26), (2.27) to get parametrizations of the sphere \(S(E_0, \rho)\) by \((\lambda, \alpha)\) and of the ball \(B(E_0, \rho)\) by \((\lambda, \alpha, s), 0 \leq s \leq \rho\).

Proposition 2.4. The longitude half circle \(\mathcal{C}(E_0, \rho; \lambda)\) with fixed \(\lambda\) and varying \(\alpha\) in \(S(E_0, \rho)\) is a smooth curve. The sphere \(S(E_0, \rho)\) is a smooth surface for any \(\rho \in \mathbb{R}_+\). All the functions and their (multiple) derivatives by \(\alpha\) in (2.25), (2.26), (2.27) are defined by limit extension at \(\alpha = \pi/4\) and at \(\alpha = -\pi/4\).

The projective machinery provides us a distance metric \(\delta(X, Y)\) by the scalar product of \(X(x^0; x^1; x^2; x^3) \in \mathcal{H}\) and \(Y(y^0; y^1; y^2; y^3) \in \mathcal{H}\) given by (2.2). Thus,
\[
\delta(X, Y) = \frac{k(\alpha)}{2} \ln \frac{-\langle X, Y \rangle + \sqrt{\langle X, X \rangle^2 - \langle X, X \rangle \langle Y, Y \rangle}}{-\langle X, Y \rangle - \sqrt{\langle X, X \rangle^2 - \langle X, X \rangle \langle Y, Y \rangle}}. \quad (2.28)
\]

The relations between parameters \(k(\alpha)\) from (2.28) and \(\alpha\) from (2.24) can be found by comparing the translation distance by Definition 2.3 with (2.28).

Proposition 2.5. The following relations hold:
\[
k(\alpha) = \frac{1}{\sqrt{\cos 2\alpha}}, \quad \text{if} \quad -\frac{\pi}{4} < \alpha < \frac{\pi}{4},
\]
and
\[
k(\alpha) = \frac{1}{\sqrt{-\cos 2\alpha}}, \quad \text{if} \quad -\frac{\pi}{2} < \alpha < -\frac{\pi}{4} \quad \text{or} \quad \frac{\pi}{4} < \alpha < \frac{\pi}{2}.
\]
Proof. Consider points $X(1; x; y; z)$ and $Y = E_0(1; 0; 0; 0)$. For $\alpha \in [0, \pi/4)$ by (2.25) with $s = \rho$ the formula (2.28) leads to

$$e^{2s/k(\alpha)} = \frac{1 + \tanh(\rho \sqrt{2\alpha})}{1 - \tanh(\rho \sqrt{2\alpha})},$$

whence $\delta/k(\alpha) = \rho \sqrt{2\alpha}$. Therefore, $\delta = \rho$ if and only if $k(\alpha) = 1/\sqrt{2\alpha}$.

Analogously, for $\alpha \in (\pi/4, \pi/2)$ or $(-\pi/2, -\pi/4)$ by (2.26) the formula (2.28) leads to

$$e^{2s/k(\alpha)} = \frac{1 + i \tanh(\rho \sqrt{2\alpha})}{1 - i \tanh(\rho \sqrt{2\alpha})},$$

whence $\delta/k(\alpha) = i \rho \sqrt{-2\alpha}$. Therefore, $\delta = \rho$ if and only if $k(\alpha) = 1/(i \sqrt{-2\alpha})$.

For $\alpha = \pi/4$ we get degeneration in above formula leading to $\delta(X, Y) = 0$. Then we take $\delta = \rho$ by formula (2.27). \qed

2.5. Infinitesimal distance

Let the arc-length-square at the origin $E_0(1; 0; 0; 0)$ be the positive definite quadratic differential form

$$(ds)^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$ (2.29)

To find similar differential form at an arbitrary point $X(x^0; x^1; x^2; x^3)$ we apply a translation from $X$ to $E_0$ by (2.10) (the pull back method):

$$(dx^0; dx^1; dx^2; dx^3) = (dx^0; dx^1; dx^2; dx^3) \cdot T^{-1} =$$

$$= (dx^0; dx^1; dx^2; dx^3) \cdot \begin{pmatrix} x^0 & -x^1 & -x^2 & -x^3 \\ x^1 & x^0 & -x^3 & x^2 \\ -x^2 & -x^3 & x^0 & -x^1 \\ -x^3 & x^2 & x^1 & x^0 \end{pmatrix} \cdot \frac{1}{\det T}.$$ (2.30)

Then at the point $X(x^0; x^1; x^2; x^3)$ we get

$$(ds)^2 = \frac{1}{[-(x^0)^2 - (x^1)^2 + (x^2)^2 + (x^3)^2]^2} \cdot \left\{ -[dx^0]x^1 + (dx^1)x^0 - (dx^2)x^3 + (dx^3)x^2]^2 + \right.$$  

$$+ [-(dx^0)x^2 - (dx^1)x^3 + (dx^2)x^0 + (dx^3)x^1]^2 +$$

$$+ [-(dx^0)x^3 + (dx^1)x^2 - (dx^2)x^1 + (dx^3)x^0]^2 \right\},$$ (2.31)

or with Euclidean coordinates $(1; x; y; z)$

$$(ds)^2 = \frac{1}{[1 + xx - yy - zz]^2} \cdot \left\{ [(dx) - (dy)z + (dz)y]^2 + 

+ [-(dx)z + (dy) + (dz)x]^2 + [(dx)y - (dy)x + (dz)]^2 \right\}.$$ (2.32)
The volume element will be, by the standard method

\[ d\text{vol} = \frac{dx\,dy\,dz}{(1 + xx - yy - zz)^2}. \]  

(2.33)

Since \(-(x^0)^2 - (x^1)^2 + (x^2)^2 + (x^3)^2 < 0\), using the projective freedom by a positive multiplier function \(c(t)\), we can assume that

\[-(x^0)^2 - (x^1)^2 + (x^2)^2 + (x^3)^2 = -1.\]

Let us introduce the so-called hyperboloid coordinates for \(\tilde{H}\) as follows

\[(x^0; x^1; x^2; x^3) = (\cosh r \cos \varphi; \cosh r \sin \varphi; \sinh r \cos(\theta - \varphi); \sinh r \sin(\theta - \varphi)),\]

(2.34)

where \((r, \theta)\) are polar coordinates of the hyperbolic base plane, and \(\varphi\) is the fibre coordinate (see [7, 9] for details). Then

\[-(x^0)^2 - (x^1)^2 + (x^2)^2 + (x^3)^2 = -\cosh^2 r + \sinh^2 r = -1\]  

(2.35)

that agrees with our assumption. The above invariant infinitesimal arc-length-square looks in these coordinates as follows:

\[(ds)^2 = (dr)^2 + \cosh^2 r \sin^2 \varphi (d\theta)^2 + [(d\varphi) + \sinh^2 r (d\theta)]^2.\]

(2.36)

The geodesic curves and balls corresponding to the above metric (2.36) have been described explicitly in [4] in terms of the hyperboloid coordinates (2.34) (see also [12]). Also, for the volume element we have

\[ d\text{vol} = \frac{1}{2} \sinh(2r) \, dr \, d\theta \, d\varphi. \]

2.6. Translation balls

Proposition 2.6. Consider \(\tilde{H}\) with geographic coordinates \((\lambda, \alpha, s)\), where \(\lambda\) is the longitude, \(-\pi < \lambda \leq \pi\), \(\alpha\) is the altitude, \(-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}\), and \(s\) is the distance. Let \(BT(E_0, \rho)\) be the translation ball centered in \(E_0(1; 0; 0; 0)\) with radius \(\rho\). If \(X(x^0; x^1; x^2; x^3)\) belongs to \(BT(E_0, \rho)\), then its coordinates are given by the following shortened formulae:

\[
\begin{pmatrix}
  x^0 \\
  x^1 \\
  x^2 \\
  x^3
\end{pmatrix} =
\begin{pmatrix}
  \cosh(s\sqrt{\cos 2\alpha}) \\
  \sin(s\sqrt{\cos 2\alpha})/\sqrt{\cos 2\alpha} \\
  \sinh(s\sqrt{\cos 2\alpha})/\sqrt{\cos 2\alpha} \\
  \sin(s\sqrt{\cos 2\alpha})/\sqrt{\cos 2\alpha}
\end{pmatrix} \sim
\begin{pmatrix}
  \sin \alpha \\
  \cos \alpha \cos \lambda \\
  \cos \alpha \sin \lambda \\
  \cos \alpha
\end{pmatrix} \tan(s\sqrt{\cos 2\alpha})
\]

(2.37)

for \(-\frac{\pi}{2} < \alpha < \frac{\pi}{2}\), and

\[
\begin{pmatrix}
  x^0 \\
  x^1 \\
  x^2 \\
  x^3
\end{pmatrix} =
\begin{pmatrix}
  \cos(s\sqrt{-\cos 2\alpha}) \\
  \sin(s\sqrt{-\cos 2\alpha})/\sqrt{-\cos 2\alpha} \\
  \sinh(s\sqrt{-\cos 2\alpha})/\sqrt{-\cos 2\alpha} \\
  \sin(s\sqrt{-\cos 2\alpha})/\sqrt{-\cos 2\alpha}
\end{pmatrix} \sim
\begin{pmatrix}
  \sin \alpha \\
  \cos \alpha \cos \lambda \\
  \cos \alpha \sin \lambda \\
  \cos \alpha
\end{pmatrix} \tan(s\sqrt{-\cos 2\alpha})
\]

(2.38)
for \(-\frac{\pi}{2} < \alpha < -\frac{\pi}{4}\) and \(\frac{\pi}{4} < \alpha < \frac{\pi}{2}\); and
\[
(x^0; x^1; x^2; x^3) = (1; s \sin \alpha; s \cos \alpha \cos \lambda; s \cos \alpha \sin \lambda)
\] (2.39)
for \(\alpha = -\frac{\pi}{4}\) or \(\alpha = \frac{\pi}{2}\).

Proof. The result follows from the formulae of translation distance (2.25), (2.26), (2.27), respectively.

□

In Fig. 3 we demonstrate the difference between geodesic balls and translation balls centered in \(E_0(1; 0; 0; 0)\) with radius \(\rho = 1\) and with radius \(\rho = 1.5\), respectively. Here the indicated values are \(\tanh(1) = 0.76159\) and \(\tanh(1.5) = 0.90514\).

2.7. Volume formula

Proposition 2.7. Consider \(\tilde{H}\) with geographic coordinates \((\lambda, \alpha, s)\), where \(\lambda\) is the longitude, \(-\pi < \lambda \leq \pi\), \(\alpha\) is the altitude, \(-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}\), and \(s\) is the distance. If \(0 \leq |\alpha| \leq \frac{\pi}{4}\) then the volume is given by
\[
\text{vol}_1 := \int_\lambda \int_\alpha \int_s \cos(\alpha) \left( \frac{\sinh \left(s \sqrt{\cos(2\alpha)}\right)}{\cos(2\alpha)} \right)^2 \, ds \, d\alpha \, d\lambda,
\] (2.40)
if \(\frac{\pi}{4} \leq |\alpha| \leq \frac{\pi}{2}\), then the volume is given by
\[
\text{vol}_2 := \int_\lambda \int_\alpha \int_s \cos(\alpha) \left( \frac{\sin \left(s \sqrt{-\cos(2\alpha)}\right)}{-\cos(2\alpha)} \right)^2 \, ds \, d\alpha \, d\lambda.
\] (2.41)
Proof. The method is standard and the result follows from the volume element (2.33) and the Jacobians by (2.37) and (2.39), respectively.

Thus, calculating volume of a region in \( \tilde{H} \) we need to divide the region in two parts according to cases \( 0 \leq |\alpha| \leq \frac{\pi}{4} \) and \( \frac{\pi}{4} \leq |\alpha| \leq \frac{\pi}{2} \) as above. Then the whole volume will be given as a sum \( \text{vol} = \text{vol}_1 + \text{vol}_2 \) of volumes of two parts. In Fig. 4 we demonstrate numerically that volumes of geodesic balls (by [4], [12]) are larger than volumes of translation balls of the same radius, both are centered in \( E_0(1; 0; 0; 0) \).

\[\text{Figure 4. Volumes for geodesic balls are larger than for translation balls.}\]

3. Prism tilings under the group \( pq2_1 \)

3.1. A family of prisms

In [22] J. Szirmai defined the bounded regular prisms and described their tilings in \( \text{SL}_2(\mathbb{R}) \) by the rotational isometry group \( pq2_1 \). This realizes the orbifold group \( G_{p,q} \) corresponding to the 2-component link \( 4^2_1 \) (the double link) pictured in Fig. 1.

The group \( G_{p,q} \) presentation is given by (1.2).

The group \( G_{p,q} = pq2_1 \) was explicitly constructed on the base of Fig. 5 by its fundamental (topological) polyhedron \( A_1, \ldots, A_p, B_1, \ldots, B_p \) with the following face pairing:

\[ a : a^{-1} := O O^p B_p(O) \cup O B_p A_1(O) \mapsto O O^p B_1(O) \cup O B_1 A_2(O) =: a, \quad (3.1) \]
Figure 5. The fundamental polyhedron in the case \( p = 3, q = 7 \)

as a \( p \)-rotation about the \( x \)-axis \( \mathbf{O}\mathbf{O}^* \) through angle \( 2\pi/p \);

\[
\mathbf{b} : \quad b^{-1} := A_1 A_2 B_1(A_1) \mapsto A_1 B_2 B_1(A_1) =: b, \quad (3.2)
\]
as a \( q \)-rotation about the fibre line \( A_1 B_1 = f_1 \), through angle \( 2\pi/q \);

\[
\mathbf{s} : \quad s^{-1} := OA_1 A_2(O) \mapsto O^* B_2 B_1(O^*) =: s, \quad (3.3)
\]
as a screw motion along \( \mathbf{O}\mathbf{O}^* \). The Poincaré algorithm (see [8] for details) will provide also defining relations by the edge equivalence classes as follows

\[
\{\mathbf{O}\mathbf{O}^*\} : \quad a^p = 1, \quad \{A_1 B_1\} : \quad b^q = 1,
\]

\[
\{OA_1, OA_2, O^* B_1, O^* B_2\} : \quad \mathbf{a}\mathbf{s}^{-1}\mathbf{s}^{-1} = 1,
\]

\[
\{A_1 A_2, A_1 B_p, A_2 B_1, B_p B_1\} : \quad \mathbf{b}\mathbf{a}\mathbf{s}\mathbf{b}^{-1} = 1.
\]

So by \( \mathbf{s} = \mathbf{b}\mathbf{a}\mathbf{b} \) we get the presentation

\[
\mathbf{pq2_1} = \langle \mathbf{a}, \mathbf{b} : \quad \mathbf{a}^p = \mathbf{b}^q = \mathbf{ababa}^{-1}\mathbf{b}^{-1}\mathbf{a}^{-1}\mathbf{b}^{-1} = 1 \rangle, \quad (3.4)
\]
as desired by matrices for \( \mathbf{a} \) an \( \mathbf{b} \) (see [22] for details).

The coordinates of the vertices \( A_1 A_2 A_3 \ldots A_p \) of the base figure and the corresponding vertices \( B_1 B_2 B_3 \ldots B_p \) of the cover face can be computed for all given parameters by

\[
tanh(OA_1) = b := \sqrt{\frac{1 - \tan \frac{\pi}{p} \tan \frac{\pi}{q}}{1 + \tan \frac{\pi}{p} \tan \frac{\pi}{q}}}. \quad (3.5)
\]

Moreover, the equation of the curve \( c_{A_1 A_2} \), connecting \( A_1 \) and \( A_2 \), can be determined (see [22] for details).
3.2. The volume of the bounded regular prisms $P_p(q)$

The volume formula of a sector-like 3-dimensional domain $\text{vol}(D(\Phi))$ can be computed by (2.36) in hyperboloid coordinates. This is defined on the base of figure $D$ lying in the base plane (see Fig. 5) and by fibre translation $\tau$ given by (2.10) with height $\Phi$ depending on $(p,q)$.

**Theorem 3.1.** Suppose we are given a sector-like region $D$ illustrated in Fig. 5, left, so a continuous function $r = r(\theta)$ where the radius $r$ depends upon the angle $\theta$. The volume of domain $D(\Phi)$ is derived by the following integral:

$$
\text{vol}(D(\Phi)) = \int_D \frac{1}{2} \sinh(2r) \, dr \, d\theta \, d\phi = 
\int_0^\Phi \int_{\theta_1}^{\theta_2} \int_{r(\theta)}^{r(\theta)} \frac{1}{2} \sinh(2r) \, dr \, d\theta \, d\phi = \Phi \int_{\theta_1}^{\theta_2} \frac{1}{4}(\cosh(2r(\theta)) - 1) \, d\theta.
$$

Let $T_p(q)$ be the regular prism tiling above and let $P_p(q)$ be one of its tiles. We get the following

**Theorem 3.2.** Let $P_p(q)$ be a bounded regular prism with integer parameters $p$ and $q$ such that $p \geq 3$ and $\frac{2p}{p+q} < q$. Then its volume can be computed by the following formula:

$$
\text{vol}(P_p(q)) = \text{vol}(D(p, q, \Phi)) \cdot p,
$$

where $\text{vol}(D(p, q, \Phi))$ is the volume of the sector-like 3-dimensional domain that is given by the sector region $OA_1A_2 \subset P$ (see Fig. 5, right) and by $\Phi = A_1B_1$, the $\widetilde{\text{SL}_2(\mathbb{R})}$ height of the prism, depending on $p$ and $q$.

3.3. The optimal translation ball packings to prism tilings $T_p(q)$ under $pq2_1$

Let $T_p(q)$ be a regular prism tiling and let $P_p(q)$ be one of its tiles which is given by its base figure that is centered at the origin with vertices $A_1A_2A_3 \ldots A_p$ in the base plane of the model (see Fig. 6) and the corresponding vertices $B_1B_2B_3 \ldots B_p$ and $C_1C_2C_3 \ldots C_p$ are generated by fibre translations $\tau := abab = baba$ and its inverse, given by (2.5) with parameter $\Phi$.

It can be assumed by symmetry arguments that the optimal translation ball is centered in the origin. Denote by $TB(E_0, \rho_{\text{opt}})$ the translation ball centered in $E_0(1; 0; 0; 0)$ of radius $\rho_{\text{opt}}$ we are searching for. The volume $\text{vol}(P_p(q))$ is given by the parameters $p$, $q$ and $\Phi \geq \rho_{\text{opt}}$. The images of $P_p(q)$ by the discrete group $pq2_1$ covers the $\widetilde{\text{SL}_2(\mathbb{R})}$ space without overlap. For the density of the packing it is sufficient to relate the volume of the optimal ball to that of the solid $P_p(q)$ (see Definition 1.2).

The optimal radius $\rho_{\text{opt}}$ can be determined as the distance between the origin and the curve $c_{A_1A_2}$, related to the half height

$$
\Phi/2 = \pi/2 - \pi/p - \pi/q
$$
of the prism, determined by $p$ and $q$. We study only one case of the multiply transitive translation ball packings where the fundamental domain of the $\tilde{\text{SL}}_2(\mathbb{R})$ space group $pq2_1$ is not a prism. Let the fundamental domains be derived by the Dirichlet — Voronoi cells (D-V cells) where their centers are images of the origin. The volume of the fundamental domain and of the D-V cell is the same, respectively, as in the prism case (for any above fixed $(p,q)$).

These locally densest translation ball packings can be determined for all possible fixed integer parameters $p$ and $q$ such that $p \geq 3$ and $\frac{2p}{p-2} < q$. The optimal radius $\rho_{\text{opt}}$ is

$$\rho_{\text{opt}} = \min \left\{ \text{artanh} \left( \frac{OA_1}{2} \right), \frac{\Phi}{2}, \frac{d(O,O^{ab})}{2} \right\},$$

(3.8)

where $d(O,O^{ab})$ is the translation distance between $O$ and $O^{ab}$ by Definition 2.3.

The maximal density of the above ball packings can be computed for any possible parameters $p, q$. In Table 1 we have summarized some numerical results. The best density that we found corresponds to the case $(p,q) = (5,10000)$ (indicating that $q \to \infty$) and is equal to 0.84170.

Our projective method gives a way to study and solve similar problems in the Thurston geometries (see e.g. [1, 10, 11, 18, 19, 20, 21, 23]). In the present paper we examined packings of $\tilde{\text{SL}}_2(\mathbb{R})$ related to the double-link groups $G_{p,q}$ action. In the forthcoming paper we will examine packings in $\tilde{\text{SL}}_2(\mathbb{R})$ with equal (geodesic and translation) balls, which are related to the trefoil orbifold groups action, described in [13]. We expect that this approach will give very dense ball packings.

References


Packings by translation balls in $\widetilde{\text{SL}}_2(\mathbb{R})$

Table 1. Results of computations.

<table>
<thead>
<tr>
<th>$(p,q)$</th>
<th>$\rho_{opt}$</th>
<th>$\text{vol}(TB(E_0, \rho_{opt}))$</th>
<th>$\text{vol}(P_p(q))$</th>
<th>$\delta(K_{opt})$</th>
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</tbody>
</table>


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