HYPERBOLIC VOLUMES OF FIBONACCI MANIFOLDS

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This article is devoted to the study of three-dimensional compact orientable hyperbolic manifolds connected with the Fibonacci groups. The Fibonacci groups

\[ F(2, m) = \langle x_1, x_2, \ldots, x_m : x_i x_{i+1} = x_{i+2}, i \mod m \rangle \]

were introduced by J. Conway [1]. The first natural question connected with these groups was whether they are finite or not [1]. It is known from [2-6] that the group \( F(2, m) \) is finite if and only if \( m = 1, 2, 3, 4, 5, 7 \). Some algebraic generalizations of the groups \( F(2, m) \) were considered in [7].

A new stage in studying the Fibonacci groups began with [5], where it was shown that the group \( F(2, 2n), n \geq 4 \), is isomorphic to a discrete cocompact subgroup of \( \text{PSL}_2(\mathbb{C}) \), the full group of orientation-preserving isometries of the Lobachevskii space \( \mathbb{H}^3 \). Moreover, the group \( F(2, 6) \) is isomorphic to a three-dimensional affine group.

The hyperbolic manifolds \( M_n = \mathbb{H}^3 / F(2, 2n), n \geq 4 \), uniformized by Fibonacci groups are referred to as the Fibonacci manifolds.

It was shown in [8] that the manifold \( M_n \) is the \( n \)-fold cyclic covering of the three-dimensional sphere \( S^3 \) branched over the figure-eight knot. We note that \( M_n \) are isometric to the hyperbolic manifolds described in [9].

In the present article we continue studying the algebraic, topological, and arithmetic properties of the Fibonacci manifolds. We establish that the hyperbolic volumes of the manifolds \( M_n \) agree with the volumes of the noncompact hyperbolic manifolds arising from complementing some well-known knots and links. In consequence it is shown that there are arithmetic and nonarithmetic manifolds with the same hyperbolic volume.

§ 1. Hyperbolic Volumes. The Thurston-Jørgensen Theorem

In this section we recall some properties of the volumes of hyperbolic manifolds. An \( n \)-dimensional hyperbolic manifold is thought of as the quotient space \( M^n = \mathbb{H}^n / \Gamma \), where \( \Gamma \) is a fixed-point-free discrete group of isometries of the Lobachevskii space \( \mathbb{H}^n \). The notions of hyperbolic area and hyperbolic volume in \( \mathbb{H}^2 \) and \( \mathbb{H}^3 \) are naturally carried over to \( M^2 \) and \( M^3 \). Further we consider the set \( \mathcal{M}^n, n = 2, 3, \) of all \( n \)-dimensional orientable hyperbolic manifolds of finite volume.

Consider the volume function \( v_n : \mathcal{M}^n \rightarrow \mathbb{R}, n = 2, 3, \) that associates the hyperbolic volume \( \text{vol}(M^n) \) with each manifold \( M^n \in \mathcal{M}^n \). It is worth observing that the volume functions \( v_2 \) and \( v_3 \) have essentially different properties.

The two-dimensional case is completely described by the Gauss-Bonnet theorem. If \( M^2 \) is a hyperbolic surface of genus \( g \) with \( k \) points removed, then

\[ \text{vol}(M^2) = 2\pi (2g - 2 + k). \]

Therefore, the range of the function \( v_2 \) is a discrete set of the form \( 2\pi \mathbb{N} \), where \( \mathbb{N} \) is the set of positive integers (see Fig. 1):

\begin{align*}
1 & \quad 2 & \quad 3 & \quad 4 \\
2\pi & \quad 4\pi & \quad 6\pi & \quad 8\pi
\end{align*}

Fig. 1
Given $v_0 = 2\pi n_0$, $n_0 \in \mathbb{N}$, there are only finitely many nonhomeomorphic surfaces $M^2$ with area $\text{vol}(M^2) = v_0$. All of them satisfy the equality

$$2g - 2 + k = n_0.$$ 

In particular, given an even $n_0 \in \mathbb{N}$, there are compact and noncompact surfaces with the same area $v_0 = 2\pi n_0$.

In the three-dimensional case the following remarkable theorem of Thurston and Jørgensen is valid: *the set of the volumes of three-dimensional hyperbolic manifolds is a well-ordered subset of type $\omega^\omega$ in the real line.* This set is plotted schematically in Fig. 2, where some well-known values of the function $v_3$ are listed.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>$\omega$</th>
<th>$\omega + \omega$</th>
<th>$\omega^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.94...</td>
<td>0.98...</td>
<td>1.01...</td>
<td>2.02...</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 2

In particular, it follows from the Thurston-Jørgensen theorem that there exists a three-dimensional hyperbolic manifold of the least volume. Some conjecture on the structure of the initial segment of the set of volumes was suggested in [10]. The manifold constructed independently by J. Weeks [11] and S. V. Matveev and A. T. Fomenko [10] has the least volume, 0.94..., among the manifolds known so far. The manifold obtained by W. Thurston [12] by the (5,1)-Dehn surgery on the figure-eight knot has the second known volume 0.98... . The third known value 1.01... is equal to the volume of the Meyerhoff-Neumann manifold [13]. We point out that this value is not on the list of [10]. The minimal manifold among known noncompact hyperbolic manifolds is the complement of the figure-eight knot. Its volume equals 2.02... and corresponds to the first limit ordinal number in the set of volumes.

In [12] W. Thurston constructed two noncompact manifolds with the different number of cusps, but with the same volume which corresponds to a limit ordinal of the set $\omega^\omega$. In the same article he posed the question of existence of a compact hyperbolic manifold whose volume corresponds to a limit ordinal. Below (see the theorem in § 5) we show that the compact Fibonacci manifolds enjoy this property.

§ 2. Fibonacci Manifolds as Branched Coverings

It was shown in [8] that the Fibonacci manifold $M_n$ can be represented as the $n$-fold cyclic covering of the three-dimensional sphere $S^3$ branched over the figure-eight knot (see Fig. 3). It means that $M_n$ is the $n$-fold covering of the orbifold $O(n)$ whose underlying space is $S^3$ and whose singular set is the figure-eight knot with index $n$.

![Fig. 3](image)

![Fig. 4](image)

The orbifold $O(n)$ has a rotational symmetry of order 2 whose set of fixed points is disjoint from the singular set of the orbifold. After factorizing by this symmetry we obtain the orbifold $6_2^2(2, n)$.
with underlying space $S^3$ and singular set the link $6_2$ (in notations of [14]) of two components with indices 2 and $n$ (see Fig. 4).

The above implies that the following diagram of coverings holds for the Fibonacci manifolds $M_n$ and the orbifolds $O(n)$ and $6_2^2(2,n)$ (see Fig. 5):

$$
\begin{array}{ccc}
M_n & \downarrow n & \\ O(n) & \downarrow 2 & \\ 6_2^2(2,n)
\end{array}
$$

Therefore, the hyperbolic volumes satisfy the relation

$$
\text{vol}(M_n) = n\text{vol}(O(n)) = 2n\text{vol}(6_2^2(2,n)).
$$

In the general case, denote by $6_2^2(m,n)$, $m,n \in \mathbb{N} \cup \{\infty\}$, the orbifold with underlying space $S^3$ and singular set the link $6_2$ of two components with indices $m$ and $n$. Observe that the orbifold $6_2^2(m,n)$ can be obtained by the generalized Dehn surgery with parameters $(m,0)$ and $(n,0)$ on the two components of the link $6_2$. The index $\infty$ indicates the removal of the corresponding component. In this case we deal with a noncompact orbifold.

Now, consider noncompact manifolds connected with the link $6_2$. Denote by $Th_n$, $n \geq 2$, the closed 3-strings braid $(\sigma_1 \sigma_2^{-1})^n$. Observe that the members of the family $Th_n$ are well known. In particular, $Th_2$ is the figure-eight knot, $Th_3$ are the Borromean rings, $Th_4$ is the Turk's head knot 818 and $Th_5$ is the knot 10123 in the notation of [14]. It was shown in [12] that the manifolds $S^3 \setminus Th_n$, $n \geq 2$, are hyperbolic and can be represented as the $n$-fold cyclic coverings of the orbifold $6_2^2(n,\infty)$. In particular, for the hyperbolic volumes we have

$$
\text{vol}(S^3 \setminus Th_n) = n\text{vol}(6_2^2(n,\infty)).
$$

The values of the volumes in (1) and (2) will be calculated in § 3 and § 4.

§ 3. Volumes of Compact Orbifolds and Manifolds

In this section, we calculate the volumes of the above-introduced compact hyperbolic orbifolds by means of the Lobachevskii function.

We recall that an ideal tetrahedron $T$ in $\mathbb{H}^3$ with four ideal vertices is described completely (up to isometry) by a single complex parameter $z$ with $\text{Im} z > 0$. In this case the dihedral angles of the tetrahedron $T = T_z$ equal $\text{arg} z$, $\text{arg} \frac{z-1}{z}$, and $\text{arg} \frac{1}{1-z}$; and each value occurs twice for a pair of opposite edges.

It is well known [15, 16] that the volume of the ideal tetrahedron $T_z$ is given by

$$
\text{vol}(T_z) = \Lambda(\text{arg} z) + \Lambda \left( \text{arg} \frac{z-1}{z} \right) + \Lambda \left( \text{arg} \frac{1}{1-z} \right),
$$

where

$$
\Lambda(x) = -\int_0^x \ln|2\sin \zeta| \, d\zeta
$$
is the Lobachevskii function. We recall some properties of the function $\Lambda(x)$:

$$
\Lambda(-x) = -\Lambda(x), \quad \Lambda(x + \pi) = \Lambda(x).
$$

Below we express the hyperbolic volumes of the orbifolds $O(n)$ and $6_2^2(2,n)$ and the manifolds $M_n$ in terms of the Lobachevskii function.
Lemma 1. For \( n \geq 4 \) the hyperbolic volume of the orbifold \( \mathcal{O}(n) \) is equal to

\[
\text{vol}(\mathcal{O}(n)) = 2(\Lambda(\beta + \delta) + \Lambda(\beta - \delta)),
\]

where \( \delta = \pi/n \) and \( \beta = 1/2 \arccos(\cos(2\delta) - 1/2) \).

**Proof.** Consider the orbifold \( \mathcal{O}(n) \) as the result of performing the generalized \((n, 0)\)-Dehn surgery to the complement of the figure-eight knot. By analogy to [17], the orbifold \( \mathcal{O}(n) \) can be obtained by the completion of the noncomplete hyperbolic structure on the union of two ideal tetrahedra \( T_z \) and \( T_w \) whose complex parameters \( z \) and \( w \) satisfy the conditions

\[
z w(z - 1)(w - 1) = 1, \quad (w(1 - z))^n = 1, \quad \text{Im} \, z > 0, \quad \text{Im} \, w > 0. \tag{4}
\]

From here we obtain the following equation in \( z \):

\[
z^2 + \left(2i \sin \frac{2\pi}{n} - 1\right)z + e^{-2\pi i/n} = 0.
\]

It has the solution

\[
z = \frac{1}{2} - i \sin \left(\frac{2\pi}{n}\right) \pm i \sqrt{1 - \left(\cos \left(\frac{2\pi}{n}\right) - \frac{1}{2}\right)^2}.
\]

Setting \( \varphi = 2\pi/n, \ n \geq 4 \), we have \(-1/2 \leq \cos \varphi - 1/2 < 1/2\). Choose \( \psi, \ 0 < \psi < \pi \), such that \( \cos \psi = \cos \varphi - 1/2 \). Then \( z = 1/2 + i(\sin \psi - \sin \varphi) \). By virtue of the condition \( \text{Im} \, z > 0 \), we choose the solution with the plus sign:

\[
z = \frac{1}{2} + i(\sin \psi - \sin \varphi). \tag{5}
\]

Therefore, from (4) we have

\[
w = \frac{\cos \varphi + i \sin \varphi}{1/2 - i(\sin \psi - \sin \varphi)}. \tag{6}
\]

For \( n \geq 5 \) expressions (5) and (6) satisfy conditions (4). In the case \( n = 4 \) we have \( \text{Im} \, z < 0 \) and \( \text{vol}(T_z) < 0 \). It means that the volume of the orbifold equals the difference of the volumes of the tetrahedra \( T_w \) and \( T_z \).

For finding the volume of the ideal tetrahedron \( T_z \) with complex parameter \( z \), we shall calculate the values of the following arguments of complex numbers:

\[
\arg z, \quad \arg \frac{z - 1}{z}, \quad \arg \frac{1}{1 - z}.
\]

**Proposition 1.** With the above notation, the following equalities hold:

\[
\arg z = \arg \frac{1}{1 - z} = \frac{\pi - \varphi - \psi}{2}, \quad \arg \frac{z - 1}{z} = \varphi + \psi.
\]

**Proof.** By straightforward computation from (5) we have

\[
\tan(\arg z) = \frac{\sin \psi - \sin \varphi}{1/2} = \frac{\sin \psi - \sin \varphi}{\cos \varphi - \cos \psi} = \cot \frac{\psi + \varphi}{2} = \tan \frac{\pi - \varphi - \psi}{2}.
\]

Similarly, for the second complex parameter we obtain

\[
\frac{1}{1 - z} = \frac{1}{1/2 - i(\sin \psi - \sin \varphi)} = \frac{1/2 + i(\sin \psi - \sin \varphi)}{1/4 + (\sin \psi - \sin \varphi)^2},
\]
\[
\tan \left( \arg \frac{1}{1-z} \right) = \sin \psi - \sin \varphi = \tan \frac{\pi - \varphi - \psi}{2}.
\]

Therefore,
\[
\arg z = \arg \frac{1}{1-z} = \frac{\pi - \varphi - \psi}{2}.
\]

To prove the remaining part of Proposition 1, observe that
\[
\arg z + \arg \frac{z-1}{z} + \arg \frac{1}{1-z} = \pi.
\]

Hence,
\[
\arg \frac{z-1}{z} = \pi - (\pi - \varphi - \psi) = \varphi + \psi,
\]

which completes the proof.

From Proposition 1 and formula (3) we infer that
\[
vol(T_z) = \Lambda(\varphi + \psi) + 2\Lambda \left( \frac{\pi - \varphi - \psi}{2} \right).
\]

Now, we turn to considering the tetrahedron \( T_w \) with complex parameter \( w \).

**Proposition 2.** With the above notation, the following equalities hold:
\[
\arg w = \arg \frac{1}{1-w} = \frac{\pi - \varphi + \psi}{2}, \quad \arg \frac{w-1}{w} = \psi - \varphi.
\]

**PROOF.** Using Proposition 1, from (6) we obtain
\[
\arg w = \arg \frac{e^{i\varphi}}{1-z} = \varphi + \frac{\pi - \varphi - \psi}{2} = \frac{\pi - \psi + \varphi}{2}.
\]

Similarly,
\[
\frac{w-1}{w} = 1 - \frac{1}{w} = 1 - \frac{1/2 - i (\sin \psi - \sin \varphi)}{\cos \varphi + i \sin \varphi} = \frac{\cos \varphi - 1/2 + i \sin \psi}{\cos \varphi + i \sin \varphi} = \frac{\cos \psi + i \sin \varphi}{\cos \varphi + i \sin \varphi} = e^{i(\psi - \varphi)},
\]

and therefore \( \arg((w-1)/w) = \psi - \varphi \). Thus,
\[
\arg \frac{1}{1-w} = \pi - \arg w - \arg \frac{w-1}{w} = \pi - \frac{\pi - \psi + \varphi}{2} - (\psi - \varphi) = \frac{\pi - \psi + \varphi}{2},
\]

which completes the proof.

From Proposition 2 and formula (3) we infer that
\[
vol(T_w) = \Lambda(\psi - \varphi) + 2\Lambda \left( \frac{\pi + \varphi - \psi}{2} \right).
\]

The volume of the orbifold \( \mathcal{O}(n) \) equals
\[
vol(\mathcal{O}(n)) = vol(T_z) + vol(T_w)
\]
\[
= 2\Lambda \left( \frac{\pi - \psi - \varphi}{2} \right) + \Lambda(\varphi + \psi) + 2\Lambda \left( \frac{\pi - \psi + \varphi}{2} \right) + \Lambda(\psi - \varphi)
\]
\[
= 2 \left( \Lambda \left( \frac{\psi + \varphi}{2} \right) + \Lambda \left( \frac{\psi - \varphi}{2} \right) \right).
\]

In the last equality we used the following property of the Lobachevskii function [16]:
\[
2\Lambda(x) = \Lambda(2x) + 2\Lambda \left( \frac{\pi}{2} - x \right).
\]

We return to the proof of Lemma 1. Assign \( \delta = \varphi/2 \) and \( \beta = \psi/2 \). Then \( \delta = \pi/n \) and \( \beta = 1/2 \arccos(\cos(2\delta) - 1/2) \). Therefore,
\[
vol(\mathcal{O}(n)) = 2(\Lambda(\beta + \delta) + \Lambda(\beta - \delta)),
\]

which completes the proof of Lemma 1.

From the diagram of coverings (Fig. 5) and Lemma 1 we obtain
Corollary 1. For \( n \geq 4 \) the hyperbolic volume of the Fibonacci manifold \( M_n \) is equal to
\[
\text{vol}(M_n) = 2n(\Lambda(\beta + \delta) + \Lambda(\beta - \delta)),
\]
where \( \delta = \pi/n \) and \( \beta = \frac{1}{2} \arccos(\cos(2\delta) - 1/2) \).

Corollary 2. For \( n \geq 4 \) the orbifold \( 6_2^2(2, n) \) is hyperbolic and
\[
\text{vol}(6_2^2(2, n)) = \Lambda(\beta + \delta) + \Lambda(\beta - \delta),
\]
where \( \delta = \pi/n \) and \( \beta = \frac{1}{2} \arccos(\cos(2\delta) - 1/2) \).

For some values of \( n \) the arguments of the Lobachevski\' function in Lemma 1 admit simpler expressions.

Corollary 3. For \( n = 4 \) the following equality holds:
\[
\text{vol}(O(4)) = \frac{3}{2} \Lambda \left( \frac{\pi}{3} \right).
\]

Proof. For \( n = 4 \) we have \( \delta = \pi/4 \), \( \beta = \pi/3 \). In this case Lemma 1 implies
\[
\text{vol}(O(4)) = 2 \left( \Lambda \left( \frac{\pi}{3} + \frac{\pi}{4} \right) + \Lambda \left( \frac{\pi}{3} - \frac{\pi}{4} \right) \right) = 2 \left( \Lambda \left( \frac{7\pi}{12} \right) + \Lambda \left( \frac{\pi}{12} \right) \right).
\]
Recall that the Lobachevski\' function has the following property [15]:
\[
\Lambda(m\theta) = m \sum_{k=0}^{m-1} \Lambda \left( \theta + \frac{k\pi}{m} \right). \tag{7}
\]
For \( m = 4 \) and \( \theta = \pi/12 \), from (7) we obtain \( \Lambda(\pi/4) = 4(\Lambda(7\pi/12) + \Lambda(\pi/3) + \Lambda(7\pi/12) - \Lambda(\pi/6)) \) by straightforward computation. For \( m = 2 \) and \( \theta = \pi/6 \), from (7) we have \( 2\Lambda(\pi/6) = 3\Lambda(\pi/3) \); hence, \( 3\Lambda(\pi/3) = 4(\Lambda(7\pi/12) + \Lambda(\pi/12)) \), which completes the proof of the corollary.

Corollary 4. For \( n = 6 \) the following equality holds:
\[
\text{vol}(O(6)) = \frac{8}{3} \Lambda \left( \frac{\pi}{4} \right).
\]

Proof. For \( n = 6 \) we have \( \delta = \pi/6 \) and \( \beta = \pi/4 \). By Lemma 1,
\[
\text{vol}(O(6)) = 2 \left( \Lambda \left( \frac{\pi}{4} + \frac{\pi}{6} \right) + \Lambda \left( \frac{\pi}{4} - \frac{\pi}{6} \right) \right) = 2 \left( \Lambda \left( \frac{5\pi}{12} \right) + \Lambda \left( \frac{\pi}{12} \right) \right).
\]
For \( m = 3 \) and \( \theta = \pi/12 \), from (7) we obtain \( 4\Lambda(\pi/4) = 3(\Lambda(5\pi/12) + \Lambda(\pi/12)) \) by straightforward computation, and the corollary follows.
A similar argument yields

Corollary 5. For \( n = 10 \) the following equality holds:
\[
\text{vol}(O(10)) = 2 \left( \Lambda \left( \frac{3\pi}{10} \right) + \Lambda \left( \frac{\pi}{10} \right) \right).
\]
§ 4. Volumes of Noncompact Orbifolds and Manifolds

To calculate the volume of the manifold $S^3 \setminus T_\infty$, we need the following

Lemma 2. For $n \geq 2$ the orbifold $6^n_2(n, \infty)$ is hyperbolic and

$$\text{vol}(6^n_2(n, \infty)) = 4(\Lambda(\alpha + \gamma) + \Lambda(\alpha - \gamma)),$$

where $\gamma = \pi/2n$ and $\alpha = 1/2 \arccos(\cos(2\gamma) - 1/2)$.

Proof. Choose generators $a$ and $\tau$ of the fundamental group $\pi_1(S^3 \setminus 6^n_2)$ in the manner indicated in Fig. 6.

Fig. 6

Using the Wirtinger algorithm [18], we obtain the following presentation for $\pi_1(S^3 \setminus 6^n_2)$:

$$\langle a, \tau \mid (\tau a^{-1} \tau a^{-1}) (\tau^{-2} a^{-1} \tau a^{-1} a^{-2}) (\tau a^{-1} \tau a^{-1} a^{-1} a^{-2})^{-1} = a \rangle.$$

With new generators $x$ and $y$ such that $a = x^{-1} y^{-1}$ and $\tau = y^{-1}$, the group has presentation

$$\pi_1(S^3 \setminus 6^n_2) = \langle x, y \mid (xy^{-1} x^{-2})(y^{-1} y^{-1} x^{-2} y)(xy^{-1} x^{-2})^{-1} = x^{-1} y^{-1} \rangle
= \langle x, y \mid y^{-1}(x^2 y x^{-1} y x^2)^{-1} y(x^2 y x^{-1} y x^2) = 1 \rangle
= \langle x, y \mid (x^2 y x^{-1} y x^2)(x^2 y x^{-1} y x^2)^{-1} y^{-1} = 1 \rangle. \quad (8)$$

Demonstrate that this group is isomorphic to a discrete group of isometries of the Lobachevskiǐ space. Consider some polyhedron in $\mathbb{H}^3$ composed of four ideal regular tetrahedra (see Fig. 7). Denote the ideal vertices of the polyhedron by $A, B, C, D, E, F$, and $\infty$. Let $u, v, t$, and $r$ be isometries of the hyperbolic space $\mathbb{H}^3$ which identify the following faces of the polyhedron pairwise:

- $u: ABE \rightarrow EDB,$
- $v: AEF \rightarrow BDC,$
- $t: AF\infty \rightarrow CD\infty,$
- $r: ABC\infty \rightarrow FED\infty.$

Let $\Gamma$ be the group generated by $u, v, t$, and $r$. By the Poincaré theorem, the complete list of relations for $\Gamma$ is as follows:

$$0: \quad u^2 = v,$$
$$1: \quad u = r v r^{-1} u r,$$
$$2: \quad t r t^{-1} r^{-1} = 1,$$
$$3: \quad r r^{-1} = 1.$$
Moreover, the ideal vertices of the polyhedron fall into two equivalence classes whose link diagrams are shown in Fig. 8.

The polygons in Fig. 8 consist of regular Euclidean triangles. Their edges are pairwise identified by Euclidean isometries. Consequently, the two cusps have a complete hyperbolic structure and the group

$$\Gamma = \langle u, v, t, r | u^2 = v, u = rvt^{-1}vr, trt^{-1}r^{-1} = 1 \rangle$$

has the polyhedron in Fig. 7 as a fundamental set in $\mathbb{H}^3$. As is easily seen from (8), the correspondence $x \rightarrow u$, $y \rightarrow r$ determines an isomorphism between the groups $\pi_1(S^3 \setminus \mathcal{6}_2)$ and $\Gamma$.

Now we turn to studying the orbifold $\mathcal{6}_2(n, \infty)$ which results from applying the generalized $(n, 0)$-Dehn surgery to one of two cusps of the hyperbolic manifold $S^3 \setminus \mathcal{6}_2$. It means that the orbifold $\mathcal{6}_2(n, \infty)$ can be obtained by completing the noncomplete hyperbolic structure on the union of four ideal tetrahedra (see Fig. 7) whose complex parameters $z_1$, $z_2$, $z_3$, and $z_4$ satisfy some system of algebraic equations. For finding these equations, consider the link diagrams of the two cusps of the manifold $S^3 \setminus \mathcal{6}_2$ (see Fig. 9 and Fig. 10, where $z' = (z-1)/z$ and $z'' = 1/(1-z)$).

Fig. 9. The generalized $(n, 0)$-surgery on the cusp of the manifold $S^3 \setminus \mathcal{6}_2$.

Fig. 10. The complete cusp of the manifold $S^3 \setminus \mathcal{6}_2$. 

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Looking at Fig. 9 and Fig. 10, we obtain the following system of equations:

\[
\begin{align*}
(z_1 - 1)z_2z_3(z_4 - 1) &= 1, \\
z_1(z_2 - 1)(z_3 - 1)z_4 &= 1, \\
(z_2(1 - z_1))^2 &= 1, \\
z_3(1 - z_1) &= 1, \\
\text{Im} z_i &> 0, \quad i = 1, 2, 3, 4.
\end{align*}
\]

(9)

Denoting \( \zeta = 1/(1 - z_1) \), from (9) we have

\[
z_1 = \frac{\zeta - 1}{\zeta}, \quad z_2 = e^{2\pi i/n} \zeta, \quad z_3 = \zeta, \quad z_4 = 1 - \frac{1}{e^{2\pi i/n} \zeta}.
\]

(10)

Furthermore, system (9) reduces to the equation

\[
\left( e^{\pi i/n} \zeta + \frac{1}{e^{\pi i/n} \zeta} - (e^{\nu i} + e^{-\nu i}) \right)^2 = 1,
\]

where \( \nu = \pi/n \). Choose \( \theta \) such that \( e^{\pi i/n} \zeta = e^{\theta i} \). Then \((2\cos \theta - 2\cos \nu)^2 = 1\), and hence \( \cos \theta = \cos \nu \pm 1/2 \). Since \( \cos \theta \leq 1 \), we choose the solution with the minus sign: \( \cos \theta = \cos \nu - 1/2 \). Substituting \( \zeta = e^{i(\theta - \nu)} \) into (10), we arrive at

\[
z_1 = 1 - 1/e^{i(\theta - \nu)}, \quad z_2 = e^{i(\theta + \nu)}, \quad z_3 = e^{i(\theta - \nu)}, \quad z_4 = 1 - 1/e^{i(\theta + \nu)}.
\]

Straightforward computation yields the following result:

**Proposition 3.** With the above notation, the following equalities hold:

(i) \( \arg z_1 = \arg \frac{z_1 - 1}{z_1} = \frac{\pi - \theta + \nu}{2}, \quad \arg \frac{1}{1 - z_1} = \theta - \nu; \)

(ii) \( \arg z_2 = \theta + \nu, \quad \arg \frac{z_2 - 1}{z_2} = \arg \frac{1}{1 - z_2} = \frac{\pi - \theta - \nu}{2}; \)

(iii) \( \arg z_3 = \theta - \nu, \quad \arg \frac{z_3 - 1}{z_3} = \arg \frac{1}{1 - z_3} = \frac{\pi - \theta + \nu}{2}; \)

(iv) \( \arg z_4 = \arg \frac{z_4 - 1}{z_4} = \frac{\pi - \theta - \nu}{2}, \quad \arg \frac{1}{1 - z_4} = \theta + \nu. \)

Since a tetrahedron in \( \mathbb{H}^3 \) is determined uniquely from its dihedral angles, we see that \( T_{z_1} = T_{z_3} \) and \( T_{z_2} = T_{z_4} \). Therefore, using (3) we conclude:

\[
\text{vol}(6^2(n, \infty)) = 2 \left( \Lambda(\theta + \nu) + \Lambda(\theta - \nu) + 2\Lambda \left( \frac{\pi - \theta - \nu}{2} \right) + 2\Lambda \left( \frac{\pi - \theta + \nu}{2} \right) \right).
\]

To complete the proof of Lemma 2, we assign \( \gamma = \nu/2 \) and \( \alpha = \theta/2 \). Then \( \gamma = \pi/2n \) and \( \alpha = 1/2 \arccos (\cos(2\gamma) - 1/2) \). Therefore, the expression for the volume of the orbifold takes the form

\[
\text{vol}(6^2(n, \infty)) = 4(\Lambda(\alpha + \gamma) + \Lambda(\alpha - \gamma)).
\]

The proof of Lemma 2 is complete.

In view of (2), we arrive at

**Corollary 6.** For \( n \geq 2 \) the volume of the noncompact hyperbolic manifold \( S^3 \setminus Th_n \) equals

\[
\text{vol}(S^3 \setminus Th_n) = 4n(\Lambda(\alpha + \gamma) + \Lambda(\alpha - \gamma)),
\]

where \( \gamma = \pi/2n \) and \( \alpha = 1/2 \arccos(\cos(2\gamma) - 1/2) \).
5. Volumes of Fibonacci Manifolds

The principal result of the present article is:

**Theorem 1.** For \( n \geq 2 \) the following equality holds

\[
\text{vol}(M_{2n}) = \text{vol}(S^3 \setminus Th_n).
\]

**Proof.** The claim is a consequence of Lemmas 1 and 2. Namely, we achieve the assertion by applying Corollary 1 to the manifold \( M_{2n}, n \geq 2 \), and Corollary 6 to the manifold \( S^3 \setminus Th_n, n \geq 2 \).

Thus, the volumes of the compact Fibonacci manifolds \( M_{2n} \) correspond to limit ordinals in the Thurston-Jørgensen theorem. In particular, the following assertions hold:

**Corollary 7.** The volume of the manifold \( M_4 \) is equal to the volume of the complement of the figure-eight knot.

**Corollary 8.** The volume of the manifold \( M_6 \) is equal to the volume of the complement of the Borromean rings.

Many properties of hyperbolic manifolds are determined by arithmeticity or nonarithmeticity of their fundamental groups [19]. As shown in [5, 8], the manifold \( M_n \) is arithmetic for \( n = 4, 5, 6, 8, 12 \) and nonarithmetic for the other values of \( n \). It is proven in [20] that the figure-eight knot \( Th_2 \) is the only arithmetic knot. Furthermore, it is known [21] that the link \( Th_3 \) of Borromean rings is arithmetic too.

**Corollary 9.** Manifolds with the same volume can be both arithmetic and nonarithmetic:

<table>
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<th>( n )</th>
<th>( M_{2n} )</th>
<th>( S^3 \setminus Th_n )</th>
</tr>
</thead>
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<tr>
<td>2</td>
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<td>arithmetic</td>
</tr>
<tr>
<td>3</td>
<td>arithmetic</td>
<td>arithmetic</td>
</tr>
<tr>
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</tr>
<tr>
<td>5</td>
<td>nonarithmetic</td>
<td>nonarithmetic</td>
</tr>
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</table>

We remark that, while discussing Corollary 9, A. Reid kindly informed the authors about the possibility of a number-theoretic approach to the construction of compact and noncompact arithmetic manifolds with the same volume.

**References**


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