Phenomenological description of the $D_{s0}^*(2317) \to D_s\pi^0$ decay

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(Received 21 August 2025; accepted 30 September 2025; published 4 November 2025)

For coupled channels D^0K^+, D^+K^0, $D_s^+\eta$, and $D_s^+\pi^0$, the S-wave scattering amplitudes are constructed taking into account the mixing of the isoscalar resonance $D_{s0}^*(2317)^+$ with nonresonance amplitudes with isospin $I = 1$. The phenomenological approach we use allows us to quite simply clear up the general structure of the $D_{s0}^*(2317)^+ \to D_s^+\pi^0$ decay amplitude violating isospin. We show that the phase of this amplitude coincides with the phase of the nonresonanct $D_s^+\pi^0$ scattering amplitude in agreement with the Watson theorem. Its modulus squared, as it should be, determines the width of the resonance peak in the $D_s^+\pi^0$ channel. Taking into account the $\pi^0 - \eta$ mixing in internal lines up to the second order inclusively ensures that the unitarity condition is fulfilled. The presented analysis complements the description of the $D_{s0}^*(2317)^+ \to D_s^+\pi^0$ decay based on the coupled channel unitarized chiral perturbation theory. The numerical estimates obtained by us for the $D_{s0}^*(2317)^+ \to D_s^+\pi^0$ decay width do not contradict those available in the literature.

DOI: 10.1103/jwkb-3zl9

I. INTRODUCTION

The isospin violating decay of the $D_{s0}^*(2317)^\pm$ state with $I(J^P) = 0(0^+)$ into $D_s^\pm\pi^0$ [1–4] has been a unique ground for fruitful theoretical studies for over 20 years [5–15]. The main sources of isotopic invariance violation in $D_{s0}^*(2317)^\pm \to D_s^\pm\pi^0$ decays are the mass differences between charged and neutral D-mesons and kaons, and the $\pi^0 - \eta$ mixing [5–15]. The most popular scheme for calculating the amplitudes of the $D_{s0}^*(2317)^+$ production with its subsequent decay into $D_s^+\pi^0$ is the coupled channel unitarized chiral perturbation theory adapted to describe the interactions of the light pseudoscalar mesons with the open charm ones, see for review [9–14,16–20] and references therein. The discovery of $D_{s0}^*(2317)^+$ was great luck for this scheme. The point is that the unknown subtraction constant present in it can be selected so that in the unitarized amplitudes under the thresholds of the coupled DK and $D_s\eta$ channels there appears a pole corresponding to the bound state with the mass of the experimentally observed $D_{s0}^*(2317)^+$ phenomenon [9–14,16–20]. Such a dynamically generated state in the system of DK and $D_s\eta$ pairs is often referred to as a hadronic molecule [14,17–20].

According to one of the latest theoretical estimates [13], the decay width $\Gamma(D_{s0}^*(2317)^+ \to D_s^+\pi^0)$ is (132 ± 7) keV of which the contribution from the $\pi^0 - \eta$ mixing accounts for (20 ± 2) keV, the contribution from the difference between the D^0K^+ and D^+K^0 loops accounts for (50 ± 3) keV, and the rest is due to constructive interference between the two isospin violation mechanisms. Although the estimates of individual contributions to $\Gamma(D_{s0}^*(2317)^+ \to D_s^+\pi^0)$ [5–15] show a noticeable scatter, they agree with each other in order of magnitude. From the experiments it is known that the total width of the $D_{s0}^*(2317)^+$ phenomenon $\Gamma < 3.8$ MeV and its decay fraction to $D_s^+\pi^0$ is $(100^{+0}_{-20})\%$ [4].

In this paper, we use a phenomenological approach to constructing the amplitudes of processes involving the $D_{s0}^*(2317)^+$ phenomenon, which has isospin $I = 0$ [4] and at the same time the only kinematically admissible hadronic decay to the channel $D_s^+\pi^0$ with $I = 1$ violating isotopic invariance. The presence of the $c\bar{s}$ pair in the quark structure of the $D_{s0}^*(2317)^+$ state indicates its possible significant couplings with the closed D^0K^+, D^+K^0, and $D_s^+\eta$ channels, as well as, due to the $\pi^0 - \eta$ mixing, with the decay channel into $D_s^+\pi^0$. The manifestation of the $D_{s0}^*(2317)^+$ in the amplitudes of the indicated channels is described by simple resonance type expressions in Sec. II. In this Section, we also construct simple expressions for the S-wave nonresonance scattering amplitudes with $I = 1$ associated with the D^0K^+, D^+K^0, and $D_s^+\pi^0$ channels, as well as, due to the $\pi^0 - \eta$ mixing, with the $D_s^+\eta$ channel. The scattering amplitudes taking into account the mixing of the isoscalar resonance $D_{s0}^*(2317)^+$ with nonresonance

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amplitudes with isospin $I = 1$ are constructed in Sec. III. The obtained formulas for the complex of mixed amplitudes with $I = 0$ and 1 allow us to determine the general structure of the amplitude of the isospin violating decay $D^*_{s0}(2317)^+ \to D^+_s \pi^0$. By the general structure we mean the main contributions that form the decay $D^*_{s0}(2317)^+ \to D^+_s \pi^0$. Using tentative values for the parameters at our disposal, we present at the end of Sec. III numerical estimates for the decay width $\Gamma(D^*_{s0}(2317)^+ \to D^+_s \pi^0)$. Brief conclusions from the analysis performed are presented in Sec. IV. Several cumbersome formulas are placed in the Appendix.

II. UNMIXED AMPLITUDES WITH $I = 0$ AND 1

A. $I = 0$ sector

Consider a state with the open charm $C = +1$, strangeness $S = +1$, isospin $I = 0$, spin-parity $J^P = 0^+$, and mass $m_r \simeq 2317.8$ MeV associated with closed D^0K^+, D^+K^0, and $D^+_s\eta$ channels (the thresholds of these channels are 2358.517 MeV, 2367.111 MeV, and 2516.212 MeV, respectively). We denote it as $D^*_{s0}(2317)^+$. In the limit of isotopic invariance, such a state is stable. However, $\pi^0 - \eta$ mixing leads to the possibility of its decay with violation of isotopic invariance to the $D^+_s\pi^0$ channel, the threshold of which is 2103.327 MeV, as well as to the possibility of $D^+_s\pi^0$ scattering via the $D^*_{s0}(2317)^+$ intermediate state

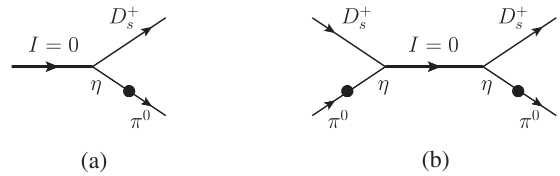

FIG. 1. (a) Decay of the $I = 0$ state, $D^*_{s0}(2317)^+$, into $D^+_s\pi^0$. (b) $D^+_s\pi^0$ scattering via the $D^*_{s0}(2317)^+$ intermediate state resulting from the $\pi^0 - \eta$ mixing. Each black circle in this figure, as well as in Figs. 2, 3, and 6, denotes the $\pi^0 - \eta$ mixing amplitude $\Pi_{\pi^0\eta}$.

[see diagrams (a) and (b) in Fig. 1]. The coupling of the isoscalar $D^*_{s0}(2317)^+$ state with the $D^+_s\pi^0$ system due to the $\pi^0 - \eta$ mixing leads to a corresponding contribution to its polarization operator. In order to correctly calculate this contribution in the second order in the amplitude of the $\pi^0 - \eta$ mixing, $\Pi_{\pi^0\eta}$, we consider the system of equations graphically depicted in Fig. 2 for the propagators of the mixed η and π^0 mesons and the propagators of the $\pi^0 \leftrightarrow \eta$ transitions denoted as $G_\eta(q^2)$, $G_\pi(q^2)$ and $G_{\eta\pi}(q^2) = G_{\pi\eta}(q^2)$, respectively. The propagators of unmixed $\tilde\eta$ and $\tilde\pi^0$ mesons are $G_{\tilde\eta}(q^2) = 1/D_{\tilde\eta}(q^2) = 1/(m^2_{\tilde\eta} - q^2)$ and $G_{\tilde\pi}(q^2) = 1/D_{\tilde\pi}(q^2) = 1/(m^2_{\tilde\pi} - q^2)$. Solving the equations in Fig. 2, we find

$$G_\eta(q^2) = \frac{D_{\tilde\pi}(q^2)}{D_{\tilde\eta}(q^2)D_{\tilde\pi}(q^2) - \Pi^2_{\pi^0\eta}}, \qquad G_\eta(q^2) = \frac{D_{\tilde\eta}(q^2)}{D_{\tilde\eta}(q^2)D_{\tilde\pi}(q^2) - \Pi^2_{\pi^0\eta}}, \qquad G_{\eta\pi}(q^2) = G_{\pi\eta}(q^2) = \frac{\Pi_{\pi^0\eta}}{D_{\tilde\eta}(q^2)D_{\tilde\pi}(q^2) - \Pi^2_{\pi^0\eta}}. \tag{1}$$

Further the easiest way to proceed is as follows. Consider, for example, for $G_\eta(q^2)$ the chain of equalities,

$$\begin{aligned} G_\eta(q^2) &= \frac{1}{D_{\tilde\eta}(q^2) - \frac{\Pi^2_{\pi^0\eta}}{D_{\tilde\pi}(q^2)}} = \frac{1}{D_\eta(q^2) - \frac{\Pi^2_{\pi^0\eta}}{D_{\tilde\pi}(q^2)} + \frac{\Pi^2_{\pi^0\eta}}{D_{\tilde\pi}(m^2_\eta)}} \\ &= \frac{1}{D_\eta(q^2)} + \frac{\Pi_{\pi^0\eta}}{D_\eta(q^2)}\left(\frac{1}{D_\pi(q^2)} - \frac{1}{D_\pi(m^2_\eta)}\right)\frac{\Pi_{\pi^0\eta}}{D_\eta(q^2)} = \frac{1}{D_\eta(q^2)} + \frac{\Pi_{\pi^0\eta}}{m^2_\eta - m^2_\pi}\left(\frac{\Pi_{\pi^0\eta}}{D_\eta(q^2)D_\pi(q^2)}\right). \end{aligned} \tag{2}$$

Here in the second equality the $\tilde\eta$ meson mass has been renormalized in the second order of perturbation theory in $\Pi_{\pi^0\eta}$ and the notation $D_\eta(q^2) = m^2_\eta - q^2$ is introduced, where m^2_η is the renormalized (physical) mass of η. Further, without violating the accuracy of the approximation, we can use the physical value for the π meson mass everywhere, i.e., replace $D_{\tilde\pi}(q^2)$ by $D_\pi(q^2) = m^2_\pi - q^2$. The third and fourth equalities are differently written first terms of the

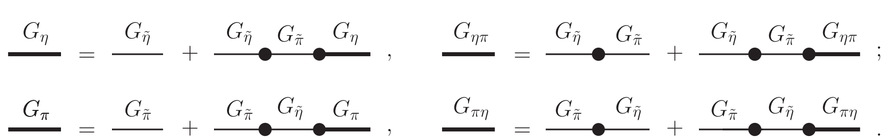

FIG. 2. Equations for propagators of the mixed η and π^0 mesons and propagators of the $\pi^0 \leftrightarrow \eta$ transitions; the argument q^2, on which the propagators depend, is omitted in the figure (the notations are explained in detail in the text).

expansion of $G_\eta(q^2)$ in $\Pi^2_{\pi^0\eta}$ (which operate within the second order of perturbation theory). Note that the expression in parentheses in the last equality is the transition propagator $G_{\eta\pi}(q^2)$ written in the first order in $\Pi_{\pi^0\eta}$. The expressions for the propagator $G_\eta(q^2)$ and similar expressions for the propagator $G_\pi(q^2)$ obtained to the second order in $\Pi_{\pi^0\eta}$ will be used below in calculating the polarization operators.

Let us proceed to the definition of the polarization operator of the $D^*_{s0}(2317)^+$ resonance. For short, we denote the channels D^0K^+, D^+K^0, $D_s^+\eta$, and $D_s^+\pi^0$ by numbers 1, 2, 3, and 4, respectively. The S-wave amplitudes $T^I_{ij}(s)$ for the reactions $i\to j$ corresponding to the contribution of the $D^*_{s0}(2317)^+$ resonance with $I=0$ are written in the form

$$T^0_{ij}(s)=\frac{g^0_ig^0_j}{D^0_r(s)}=\frac{g^0_ig^0_j}{m_r^2-s+\mathrm{Re}\Pi^0_r(m_r^2)-\Pi^0_r(s)},\qquad(3)$$

where i and j are the channel numbers, s is the invariant mass squared in channel i and j, $1/D^0_r(s)$ is the propagator of the $D^*_{s0}(2317)^+$ resonance, g^0_i is its coupling constant with channel i, in so doing $g^0_4=\epsilon g^0_3$, and $\epsilon=\Pi_{\pi^0\eta}/(m^2_\eta-m^2_{\pi^0})$ is the parameter of the $\pi^0-\eta$ mixing. Here we will be guided on the value of $\epsilon\simeq -0.014$ [21,22]. The polarization operator $\Pi^0_r(s)$ in (3) is the sum of the self-energy parts of the resonance $D^*_{s0}(2317)^+$ due to the D^0K^+, D^+K^0, $D_s^+\eta$, and $D_s^+\pi^0$-intermediate states, i.e.,

$$\Pi^0_r(s)=\sum_{i=1,2,3}\frac{(g^0_i)^2}{16\pi}G_{ii}(s)+\frac{(g^0_3)^2}{16\pi}\ddot{G}_{44}(s),\qquad(4)$$

where the functions $G_{ii}(s)$ for $i=1,2,3$ are defined as the dispersion two-body loop integrals subtracted at the threshold of the ith channel. Explicit expressions for them are written out in the Appendix. The two points in the notation of the function $\ddot{G}_{44}(s)$ indicate that we are dealing with the second-order contribution in the amplitude $\Pi_{\pi^0\eta}$. According to the last two equalities in Eq. (2), the function $\ddot{G}_{44}(s)$ can be calculated using the diagrams shown in Fig. 3 in two equivalent ways, i.e., using either the right-hand or the left-hand side of the equality shown in the figure. We write the amplitude of the right-hand side of this equality in the form $\dot{G}_{34}(s)\frac{\Pi_{\pi^0\eta}}{m^2_\eta-m^2_{\pi^0}}$, where $\dot{G}_{34}(s)$ denotes the first order in the $\Pi_{\pi^0\eta}$ convergent loop diagram, diagram (c), shown in the figure. An explicit expression for $\dot{G}_{34}(s)$ is given in the Appendix; $\dot{G}_{34}(s)=\dot{G}_{43}(s)$. Thus, $\ddot{G}_{44}(s)$ can be calculated by the rule $\ddot{G}_{44}(s)=\dot{G}_{34}(s)\frac{\Pi_{\pi^0\eta}}{m^2_\eta-m^2_{\pi^0}}$. Calculating $\ddot{G}_{44}(s)$ using diagrams (a) and (b), which define the left-hand side of the equality in Fig. 3, turns out to be more complicated. It is worth noting that at the $D_s^+\eta$ threshold, the imaginary parts of diagrams (a) and (b) in Fig. 3 have root threshold singularities, i.e., they contain contributions $\sim 1/\sqrt{s-(m_{D_s^+}+m_\eta)^2}$ that cancel out in the difference of these diagrams.

The decay width of the resonance, $\Gamma_r(s)$, is determined by the imaginary part of the polarization operator. In this case, in the region of 2317 MeV, $\sqrt{s}\Gamma_r(s)=\mathrm{Im}\,\Pi^0_r(s)=\frac{(g^0_3)^2}{16\pi}\mathrm{Im}\,\ddot{G}_{44}(s)=\frac{(g^0_3)^2}{16\pi}\frac{\Pi_{\pi^0\eta}}{m^2_\eta-m^2_{\pi^0}}\mathrm{Im}\,\dot{G}_{34}(s)$ and therefore, due to the mechanism of the $\pi^0-\eta$ mixing, we have (see Fig. 3 and the Appendix)

$$\sqrt{s}\Gamma_r(s)=\frac{(g^0_3)^2}{16\pi}\left(\frac{\Pi_{\pi^0\eta}}{m^2_\eta-m^2_{\pi^0}}\right)^2\rho_{D_s^+\pi^0}(s),\qquad(5)$$

where $\rho_{D_s^+\pi^0}(s)=\sqrt{[s-(m_{D_s^+}+m_{\pi^0})^2][s-(m_{D_s^+}-m_{\pi^0})^2]}/s$. Diagram (a) in Fig. 1 leads to exactly the same expression for $\sqrt{s}\Gamma_r(s)$ (in full agreement with the unitarity condition). Thus, we have before us a kind of "classical" resonance, which could exist in the case of the absence (or extreme smallness) of strong interactions in the D^0K^+, D^+K^0, and $D_s^+\pi^0$ channels with isospin $I=1$. Literally, such a situation seems unlikely. Consideration at this stage of the $D^*_{s0}(2317)^+$ meson as an isolated state with $I=0$ suggests that the amplitudes due to its contribution will be used to construct a more realistic picture of interactions in the coupled channels under consideration.

Let us now introduce some new notations that will be useful later. We write the coupling constants g^0_i as $g^0_i=g^0_1\eta^0_i$, where $\eta^0_i=g^0_i/g^0_1$ are dimensionless constants on the scale of g^0_1, and define the reduced (dimensionless) propagator of the $D^*_{s0}(2317)^+$ resonance $1/\tilde{D}^0_r(s)=1/[D^0_r(s)/(g^0_1)^2]$. In these notations $T^0_{ij}(s)=\eta^0_i\eta^0_j\tilde{D}^0_r(s)$. Due to isotopic invariance, $\eta^0_1=\eta^0_2=1$ and, by definition

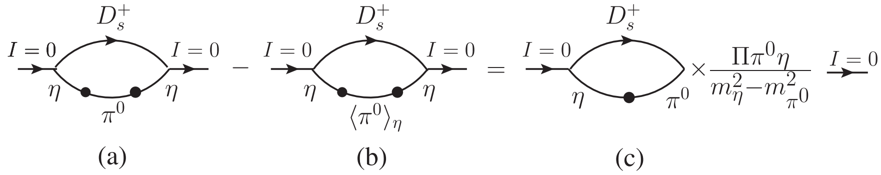

FIG. 3. According to Eq. (2) there are two equivalent ways of calculating the function $\ddot{G}_{44}(s)$ using the right-hand or left-hand side of the equality shown in the figure. In diagram (b), $\langle\pi^0\rangle_\eta$ denotes that the pion proragator $1/D_\pi(q^2)$ is taken at $q^2=m^2_\eta$.

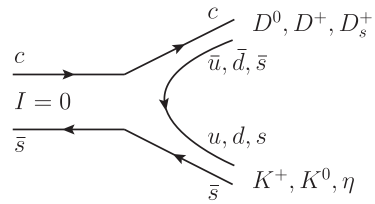

FIG. 4. Coupling of the $c\bar{s}$ state with channels D^0K^+, D^+K^0, and $D_s^+\eta$.

(see above), $\eta_4^0 = \epsilon\eta_3^0$. The value of η_3^0 can be found within the framework of a specific model of the interaction of the isoscalar $D_{s0}^*(2317)^+$ state with the channels D^0K^+, D^+K^0, and $D_s^+\eta$. For example, in the $c\bar{s}$ model, the relative values of the coupling constants $D_{s0}^*(2317)^+$ with the channels D^0K^+, D^+K^0, and $D_s^+\eta$ are easily determined using the diagram in Fig. 4. The usual quark counting rule, together with the representation of the final $s\bar{s}$ pair in terms of physical states η and η' with defined masses ($s\bar{s} = -\eta\cos\vartheta + \eta'\sin\vartheta$, where $\vartheta = \theta_i - \theta_p$, $\theta_i = 35.3°$ is the so-called ideal mixing angle and $\theta_p = -11.3°$ is the mixing angle in the nonet of light pseudoscalar mesons [4]), leads to the relations $\eta_1^0 : \eta_2^0 : \eta_3^0 = 1 : 1 : -\cos\vartheta = 1 : 1 : -0.687$. If all η_i^0 are known, then an estimate of the overall constant $(g_1^0)^2$ can be obtained using the available (so far only theoretical) values of the DK scattering length, $a_{DK}^{(0)}$, in the channel with $I = 0$ [10,19,23–27]. In accordance with the calculations [10,19,23–27] we will be guided by $a_{DK}^{(0)} \approx -1$ fm. Let us express $a_{DK}^{(0)}$ through the amplitude $T_{11}^0(s) + T_{12}^0(s) = 2T_{11}^0(s)$ at the D^0K^+ threshold. In our normalization we have[1]

$$a_{DK}^{(0)} = \frac{2T_{11}^0(s)}{8\pi\sqrt{s}}\bigg|_{\sqrt{s}=(m_{D^0}+m_{K^+})} \approx -1 \text{ fm}. \quad (6)$$

From here we find $(g_1^0)^2/(16\pi) \approx 1.884$ GeV^2 and from Eq. (5) obtain $\Gamma_r(m_r^2) \approx 19.3$ keV. In estimating $(g_1^0)^2/(16\pi)$, we neglected the negligible effect of the $\pi^0 - \eta$ mixing. Thus, we have completely determined the resonant amplitudes $T_{ij}^0(s)$.

B. $I = 1$ sector

Let us proceed to the construction of S-wave nonresonance amplitudes of the reactions $i \to j$ with isospin $I = 1$ in the s-channel $T_{ij}^1(s)$. As a guide, we will use the Zachariazen field theoretical model for the single-channel S-wave scattering amplitude that was once thoroughly investigated in the works [28–31]. The amplitude $T(s)$ in the Zachariasen model exactly coincides with the result of the summing up of all chains of s-channel loop diagrams in the theory with Lagrangian $\mathcal{L} = -\lambda_0\varphi_a^4$, where λ_0 is the seed coupling constant and φ_a is the scalar field with mass m_a [28,30,31]. In terms of renormalized quantities the amplitude $T(s) = -\lambda/[1 + \frac{\lambda}{16\pi}G(s)]$, where λ is the coupling constant and $G(s)$ is the dispersion loop integral once subtracted (for definiteness) at $s = 4m_a^2$. For positive λ, this model produces a "dynamic" bound state [29,30]. But for negative values of λ, the model gives a good example of the nonresonance scattering amplitude and phase [29,30]. We will use one of the simplest generalizations of such an amplitude for the case of several coupled channels with $I = 1$, which looks like this:

$$T_{ij}^1(s) = \frac{-\lambda\eta_i^1\eta_j^1}{D_n^1(s)} = \frac{-\lambda\eta_i^1\eta_j^1}{1 + \Pi_n^1(s)}, \quad (7)$$

where the index n indicates that the corresponding quantities describe the nonresonance case, and the polarization operator $\Pi_n^1(s)$ has the form

$$\Pi_n^1(s) = \frac{\lambda}{16\pi}\left[\sum_{i=1,2,4}(\eta_i^1)^2 G_{ii}(s) + (\eta_4^1)^2\ddot{G}_{33}(s)\right]. \quad (8)$$

Here $G_{11}(s)$, $G_{22}(s)$, and $G_{44}(s)$ are the dispersion loop integrals subtracted at the corresponding thresholds and the function $\ddot{G}_{33}(s) = -\ddot{G}_{44}(s) = -\dot{G}_{34}(s)\frac{\Pi_{\pi^0\eta}}{m_\eta^2 - m_{\pi^0}^2}$ (see the Appendix); λ is the coupling constant; $\eta_1^1 = -\eta_2^1 = 1/\sqrt{2}$, $\eta_4^1 = 1$; in (7), $\eta_3^1 = -\epsilon\eta_4^1$. Let us explain the details of such a representation of $T_{ij}^1(s)$. In the multichannel case, the seed constant λ_0 is replaced by a symmetric matrix λ_{0ij} composed of seed amplitudes of the transitions $i \to j$. Due to isotopic invariance, this matrix has the form

$$\lambda_{0ij} = \begin{pmatrix} \lambda_{011} & -\lambda_{011} & \lambda_{041} \\ -\lambda_{011} & \lambda_{011} & -\lambda_{041} \\ \lambda_{041} & -\lambda_{041} & \lambda_{044} \end{pmatrix}. \quad (9)$$

It contains three independent constants λ_{011}, λ_{041}, and λ_{044} corresponding to the transitions $D^0K^+ \to D^0K^+$, $D_s^+\pi^0 \to D^0K^+$, and $D_s^+\pi^0 \to D_s^+\pi^0$, respectively. Note that in the sector with $C = S = +1$ and $I = 1$ all intermediate states must be at least four-quark ones. We assume that the seed interaction between the continuous spectrum states D^0K^+, D^+K^0, and $D_s^+\pi^0$ is carried out due to the rearrangement of valence quarks in colliding particles, see Fig. 5. In this case, the matrix λ_{0ij} has a very simple form

[1] Note that for the given mass of the bound state m_r, the possible values of $a_{DK}^{(0)} < 0$ (due to its contribution) are limited from below by the value -1.432 fm, which is obtained if in $T_{11}^0((m_{D^0} + m_{K^+})^2)$ we let $(g_1^0)^2$ tend to infinity.

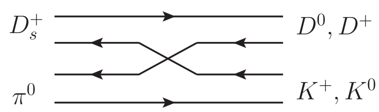

FIG. 5. Example diagram of the quark rearrangement during scattering.

$$\lambda_{0ij} = \begin{pmatrix} 0 & 0 & \lambda_0 \\ 0 & 0 & -\lambda_0 \\ \lambda_0 & -\lambda_0 & 0 \end{pmatrix} \tag{10}$$

since the rearrangement mechanism allows only the transitions $D_s^+\pi^0 \leftrightarrow D^0K^+$ and $D_s^+\pi^0 \leftrightarrow D^+K^0$ the amplitudes of which due to isotopic invariance differ in sign. Nevertheless, expressions for the amplitudes $T^1_{ij}(s)$ in the original unsubtracted form are far from trivial. In matrix notation, the result of summing up all chains of the s channel loop diagrams in the model with a seed interaction from Eq. (10) has the form $\hat{T}^1(s) = -[\hat{I} + \hat{\lambda}_0\hat{\Delta}(s)]^{-1}\hat{\lambda}_0$, where $\hat{\Delta}(s) = B\hat{I} + \hat{G}(s)$, B is an infinite constant, and $\hat{G}$ is the diagonal matrix of the dispersion loop integrals subtracted at the corresponding thresholds [30]. Renormalization in the amplitudes $T^1_{ij}(s)$ is carried out using the relation $B(1 - \frac{1}{2B^2\lambda_0^2}) = 1/\lambda$, where $B \to \infty$, $\lambda_0 \to 0$, but so that λ is a finite value. After taking into account the $\pi^0 - \eta$ mixing [similarly to how it was done in the case of the amplitudes $T^0_{ij}(s)$] we obtain Eqs. (7) and (8). We also introduce the reduced propagator $1/\tilde{D}^1_n(s) = 1/[D^1_n(s)/(-\lambda)]$. In these terms $T^1_{ij}(s) = \eta^1_i\eta^1_j/\tilde{D}^1_n(s)$.

Thus, we have constructed the resonance amplitudes $T^0_{ij}(s)$ in the sector with isospin $I = 0$ and the nonresonance ones $T^1_{ij}(s)$ in the sector with isospin $I = 1$, which do not yet know anything about each other. Now we are ready to move on to considering the mixing of these amplitudes.

III. MIXED $D^*_{s0}(2317)^+$ STATE

The amplitudes T_{ij} that would take into account the mixing of the isoscalar resonance $D^*_{s0}(2317)^+$ with the nonresonance amplitudes with isospin $I = 1$ can be constructed in the same way as was done when considering phenomena of the $a_0(980) - f_0(980)$ mixing [32]. Graphically, the scheme for accounting for mixing in the form of equations for the reduced propagators dressed by mixing $\tilde{\mathcal{G}}_0(s)$ and $\tilde{\mathcal{G}}_1(s)$ in the channels with $I = 0$ and 1, respectively, and for the propagators of the transition between these channels $\tilde{\mathcal{G}}_{01}(s)$ and $\tilde{\mathcal{G}}_{10}(s)$ looks completely similar to the scheme for accounting for mixing shown in Fig. 2. From the corresponding equations we find that the propagators dressed by mixing have the form

$$\tilde{\mathcal{G}}_0(s) = \frac{\tilde{D}^1_n(s)}{\tilde{D}^0_r(s)\tilde{D}^1_n(s) - \tilde{\Pi}^2_{01}(s)}, \qquad \tilde{\mathcal{G}}_1(s) = \frac{\tilde{D}^0_r(s)}{\tilde{D}^0_r(s)\tilde{D}^1_n(s) - \tilde{\Pi}^2_{01}(s)}, \qquad \tilde{\mathcal{G}}_{01}(s) = \tilde{\mathcal{G}}_{10}(s) = \frac{\tilde{\Pi}_{01}(s)}{\tilde{D}^0_r(s)\tilde{D}^1_n(s) - \tilde{\Pi}^2_{01}(s)}. \tag{11}$$

Here the functions $\tilde{\Pi}_{01}(s)$ and $\tilde{\Pi}_{10}(s)$ are polarization operators that are nondiagonal in isotopic spin, responsible for mixing channels with $I = 0$ and 1; $\tilde{\Pi}_{01}(s) = \tilde{\Pi}_{10}(s)$. The polarization operator $\tilde{\Pi}_{01}(s)$ has the form

$$\begin{aligned}\tilde{\Pi}_{01}(s) &= \frac{1}{16\pi}\left[\eta^0_3\dot{G}_{34}(s)\eta^1_4 + \eta^0_1\hat{G}_{11}(s)\eta^1_1 + \eta^0_1\hat{G}_{22}(s)\eta^1_2\right] \\ &= \frac{1}{16\pi}\left[-\cos\vartheta\dot{G}_{34}(s) + \frac{1}{\sqrt{2}}\hat{G}_{11}(s) - \frac{1}{\sqrt{2}}\hat{G}_{22}(s)\right].\end{aligned} \tag{12}$$

It is formed by two main sources of isotopic invariance violation, which are the $\pi^0 - \eta$ mixing and the presence of the mass differences between charged and neutral D mesons and kaons. Thanks to the latter, the contributions of the D^0K^+ and D^+K^0 intermediate states, entering to $\tilde{\Pi}_{01}(s)$ with opposite signs, do not completely compensate each other. The intermediate states in $\tilde{\Pi}_{01}(s)$ do not have a definite isotopic spin and therefore contribute to mixing. The contributions to $\tilde{\Pi}_{01}(s)$ are described by convergent expressions. The function $\dot{G}_{34}(s)$ has already been encountered in Sec. II (see, in particular, Fig. 3). In $\tilde{\Pi}_{01}(s)$, it is responsible for the contribution caused by the $\pi^0 - \eta$ mixing. The difference between the D^0K^+ and D^+K^0 loops $\hat{G}_{11}(s) - \hat{G}_{22}(s)$ [see the second and third terms in Eq. (12)] also does not contain divergence. In so doing, $\hat{G}_{11}(s) - \hat{G}_{22}(s)$ differs from the difference of the integrals $G_{11}(s)$ and $G_{22}(s)$, subtracted at the corresponding thresholds, by a constant depending on the ratios of the masses of the particles in the loops (see the Appendix).

Using the propagators $\tilde{\mathcal{G}}(s)$ from Eq. (11), it is easy to construct the amplitudes T_{ij}. The result will be exactly the same as when opening the expression

$$T_{ij} = \eta^I_i \begin{pmatrix} \tilde{D}^0_r(s) & -\tilde{\Pi}_{01}(s) \\ -\tilde{\Pi}_{10}(s) & \tilde{D}^1_n(s) \end{pmatrix}^{-1}_{I,I'} \eta^{I'}_j. \tag{13}$$

Thus we get

$$T_{ij} = \frac{\eta_i^0 \tilde{D}_n^1(s)\eta_j^0 + \eta_i^0 \tilde{\Pi}_{01}(s)\eta_j^1 + \eta_i^1 \tilde{\Pi}_{10}(s)\eta_j^0 + \eta_i^1 \tilde{D}_r^0(s)\eta_j^1}{\tilde{D}_r^0(s)\tilde{D}_n^1(s) - \tilde{\Pi}_{01}^2(s)}. \tag{14}$$

Let us consider in detail the amplitude $T_{44}(s) \equiv T_{D_s^+\pi^0 \to D_s^+\pi^0}(s)$ corresponding to the only opened hadronic decay channel $D_s^+\pi^0$ in the region of $D_{s0}^*(2317)^+$ resonance, and represent it as a sum of *"background + resonance"*:

$$\begin{aligned} T_{44} &= \frac{\eta_4^0 \tilde{D}_n^1(s)\eta_4^0 + \eta_4^0 \tilde{\Pi}_{01}(s)\eta_4^1 + \eta_4^1 \tilde{\Pi}_{10}(s)\eta_4^0 + \eta_4^1 \tilde{D}_r^0(s)\eta_4^1}{\tilde{D}_r^0(s)\tilde{D}_n^1(s) - \tilde{\Pi}_{01}^2(s)} \\ &= \frac{\eta_4^1\eta_4^1}{\tilde{D}_n^1(s)} + \frac{\left(\eta_4^0 + \frac{\tilde{\Pi}_{01}(s)\eta_4^1}{\tilde{D}_n^1(s)}\right)^2}{\tilde{D}_r^0(s) - \frac{\tilde{\Pi}_{01}^2(s)}{\tilde{D}_n^1(s)}} = \frac{16\pi}{\rho_{D_s^+\pi^0}(s)}\left\{\frac{e^{2i\delta_0^1(s)} - 1}{2i} + e^{2i\delta_0^1(s)} T_{\mathrm{res}}(s)\right\}. \end{aligned} \tag{15}$$

The first term in the second equality in Eq. (15) is the non-resonance amplitude $T_{44}^1(s) = \eta_4^1\eta_4^1/\tilde{D}_n^1(s) = 1/\tilde{D}_n^1(s) = \frac{16\pi}{\rho_{D_s^+\pi^0}(s)}[-\operatorname{Im}\tilde{D}_n^1(s)/\tilde{D}_n^1(s)] = \frac{16\pi}{\rho_{D_s^+\pi^0}(s)} e^{i\delta_0^1(s)} \sin\delta_0^1(s)$ or the background one, and $\delta_0^1(s)$ is the background phase. The expression $\eta_4^0 + \frac{\tilde{\Pi}_{01}(s)\eta_4^1}{\tilde{D}_n^1(s)}$, standing in brackets in the numerator of the second term in the second equality in Eq. (15), is the amplitude of the decay of the isoscalar resonance $D_{s0}^*(2317)^+$ (dressed by the background) into $D_s^+\pi^0$ caused by the isospin breaking. We denote it as $V_{\mathrm{res}}(s) = \eta_4^0 + \frac{\tilde{\Pi}_{01}(s)\eta_4^1}{\tilde{D}_n^1(s)}$. Here the first term $\eta_4^0 = \epsilon\eta_3^0$ is due to the contribution of the first diagram in Fig. 6 and the second one is due to the contributions of the subsequent three diagrams in this figure in accordance with Eq. (12) for the polarization operator $\tilde{\Pi}_{01}(s)$ multiplied by the amplitude $\frac{\eta_4^1}{\tilde{D}_n^1(s)}$ of the nonresonant final state interaction in the channel with $I = 1$. Using Eq. (7), the relation $\operatorname{Im}\dot{G}_{34}(s) = \epsilon\rho_{D_s^+\pi^0}(s)$ (see Appendix), the above values of η_i^I, and the relation of $1/\tilde{D}_n^1(s)$ with the phase $\delta_0^1(s)$, we obtain the following expression for $V_{\mathrm{res}}(s)$:

$$V_{\mathrm{res}}(s) = e^{i\delta_0^1(s)}\left\{-\epsilon\cos\vartheta\cos\delta_0^1(s) + \left[-\cos\vartheta\operatorname{Re}\dot{G}_{34}(s) + \frac{1}{\sqrt{2}}\left(\hat{G}_{11}(s) - \hat{G}_{22}(s)\right)\right]\frac{\sin\delta_0^1(s)}{\rho_{D_s^+\pi^0}(s)}\right\} = e^{i\delta_0^1(s)} v_{\mathrm{res}}(s). \tag{16}$$

As can be seen, the phase of the decay amplitude $V_{\mathrm{res}}(s)$ is determined by the phase of the nonresonance $D_s^+\pi^0$ scattering amplitude in accordance with the Watson theorem on the final state interaction [33] (or with the unitarity requirement). In the normalization we use, the $D_{s0}^*(2317)^+ \to D_s^+\pi^0$ decay width and the amplitude $V_{\mathrm{res}}(s)$ are connected by the relation

$$\frac{\sqrt{s}\Gamma\mathrm{res}(s)}{(g_1^0)^2} = \frac{|V_{\mathrm{res}}(s)|^2}{16\pi}\rho_{D_s^+\pi^0}(s). \tag{17}$$

It is easy to show that the imaginary part of the expression $\tilde{D}_r^0(s) - \frac{\tilde{\Pi}_{01}^2(s)}{\tilde{D}_n^1(s)}$ [this is the denominator of the second term in the second equality in Eq. (15)] is exactly equal to $\frac{|V_{\mathrm{res}}(s)|^2}{16\pi}\rho_{D_s^+\pi^0}(s)$ and, therefore, determines the width of the resonance peak in the $D_s^+\pi^0$ channel. The expression $1/[\tilde{D}_r^0(s) - \frac{\tilde{\Pi}_{01}^2(s)}{\tilde{D}_n^1(s)}]$ can naturally be called the reduced propagator of the $D_{s0}^*(2317)^+$ resonance modified by the background. The ratio $\frac{\tilde{\Pi}_{01}^2(s)}{\tilde{D}_n^1(s)} \equiv \tilde{\Pi}_{01}(s)\frac{1}{\tilde{D}_n^1(s)}\tilde{\Pi}_{10}(s)$ included in it is a polarization operator of the isoscalar state taking into account the isovector interaction between particles in the loops. Its real part at $\sqrt{s} = M_{\mathrm{res}}$, where M_{res} is the mass of the resonance dressed by the background, together with the contribution $\frac{(\eta_3^0)^2}{16\pi}\operatorname{Re}\ddot{G}_{44}(M_{\mathrm{res}}^2)$ [see Eqs. (3) and (4)], determines the magnitude of the mass shift due to isospin breaking (isotopic mass shift). Finally, taking into account Eq. (3), we write out an expression for the resonance amplitude $T_{\mathrm{res}}(s)$ in Eq. (15)

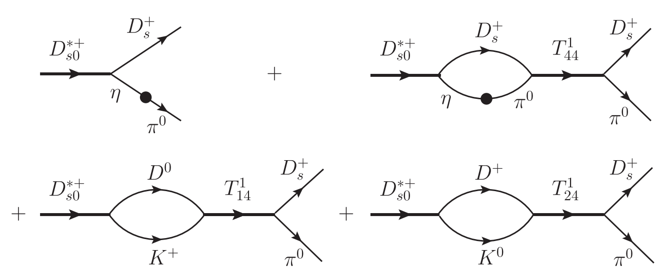

FIG. 6. Diagrams contributing to the decay amplitude $D_{s0}^*(2317)^+ \to D_s^+\pi^0$.

$$T_{\mathrm{res}}(s) = \frac{\sqrt{s}\Gamma_{\mathrm{res}}(s)}{M_{\mathrm{res}}^2 - s + \mathrm{Re}\Pi_{\mathrm{res}}(M_{\mathrm{res}}^2) - \Pi_{\mathrm{res}}(s)} = \frac{\sqrt{s}\Gamma_{\mathrm{res}}(s)}{D_{\mathrm{res}}(s)}, \tag{18}$$

where

$$\Pi_{\mathrm{res}}(s) = \Pi_r^0(s) + g_1^0 \frac{\tilde{\Pi}_{01}(s)\tilde{\Pi}_{10}(s)}{\tilde{D}_n^1(s)} g_1^0. \tag{19}$$

If $D_{s0}^*(2317)^+$ is produced by a source with isospin $I=0$, then the amplitude of its production, propagation, and decay into $D_s^+\pi^0$ can be written as

$$T_{\mathrm{prod}}(s) = \Lambda(s) \frac{e^{i\delta_0^1(s)}\xi\sqrt{\sqrt{s}\Gamma_{\mathrm{res}}(s)}}{M_{\mathrm{res}}^2 - s + \mathrm{Re}\,\Pi_{\mathrm{res}}(M_{\mathrm{res}}^2) - \Pi_{\mathrm{res}}(s)}, \tag{20}$$

where $\Lambda(s)$ is the source amplitude, $e^{i\delta_0^1(s)}$ is the phase of the nonresonance $D_s^+\pi^0$ interaction in the final state, and $\xi = v_{\mathrm{res}}(s)/|v_{\mathrm{res}}(s)|$ is the sign of the amplitude $v_{\mathrm{res}}(s)$, see (16). Formula (20) follows from Eq. (14) when taking into account in the numerator of the latter the first two terms corresponding to the source with $I=0$.

Let us return to Eq. (16) for the amplitude $V_{\mathrm{res}}(s)$ and give the numerical values of the terms forming it at $\sqrt{s} = M\mathrm{res} \simeq 2317.8$ MeV [the values of $V_{\mathrm{res}}(s)$ and $\delta_0^1(s)$ at this point we denote as $\bar{V}_{\mathrm{res}}$ and $\bar{\delta}_0^1$, respectively]:

$$\bar{V}_{\mathrm{res}} = e^{i\bar{\delta}_0^1}\left\{0.009618\cos\bar{\delta}_0^1 + [0.001888 + 0.0101372]\frac{\sin\bar{\delta}_0^1}{0.257048}\right\} = e^{i\bar{\delta}_0^1}\left(0.009618\cos\bar{\delta}_0^1 + 0.0467801\sin\bar{\delta}_0^1\right). \tag{21}$$

The value of $\bar{\delta}_0^1$ is unknown. But we can trace how the decay width of $D_{s0}^*(2317)^+ \to D_s^+\pi^0$ changes depending on $\bar{\delta}_0^1$. The corresponding picture is shown in Fig. 7. Although large values of $\bar{\delta}_0^1$ are almost improbable, Eq. (21) formally allows us to set an upper limit for $\Gamma_{\mathrm{res}}(M_{\mathrm{res}}^2)$. It is 476.5 keV at $\bar{\delta}_0^1 = 78.4°$, see Fig. 7. The values of the background phase $\bar{\delta}_0^1 \approx (15–20)°$ can be considered quite reasonable. They lead to $\Gamma_{\mathrm{res}}(M_{\mathrm{res}}^2) \approx$ (95–130) keV (see Fig. 7). These values agree very well with the calculations of $\Gamma_{\mathrm{res}}(M_{\mathrm{res}}^2)$ based on the coupled channel unitarized chiral perturbation theory and lattice QCD [9–14]. Note that the isotopic mass shift of the $D_{s0}^*(2317)^+$ resonance at $\bar{\delta}_0^1 = 15°$ and 20° amounts to −56 keV and −67 keV, respectively.

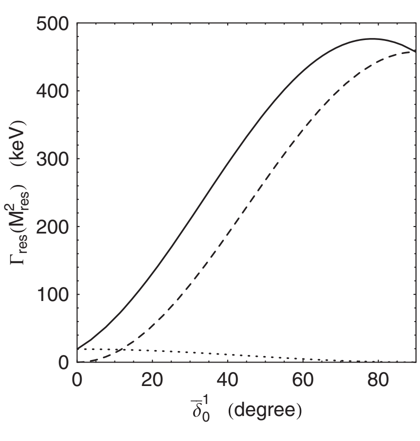

FIG. 7. Solid curve shows the decay width of $D_{s0}^*(2317)^+ \to D_s^+\pi^0$ at $\sqrt{s} = M_{\mathrm{res}} \simeq 2317.8$ MeV as a function of $\bar{\delta}_0^1$, constructed using Eqs. (21) and (17) at $(g_1^0)^2/(16\pi) = 1.884$ GeV^2. The dotted and dashed curves show the contributions due to the $\pi^0-\eta$ mixing (proportional to $\cos\bar{\delta}_0^1$) and meson loops (proportional to $\sin\bar{\delta}_0^1$), respectively; see Eq. (16).

IV. CONCLUSION

From the above analysis we draw the following conclusions.

(1) The effect of isotopic invariance violation for the $D_{s0}^*(2317)^+$ resonance is in many ways similar to the well-known threshold phenomenon of the $a_0(980)^0$ and $f_0(980)$ resonance mixing [32].

(2) The phenomenological approach we used allowed us to quite easily determine the general structure of the amplitude of the isospin violating decay of $D_{s0}^*(2317)^+ \to D_s^+\pi^0$ and the amplitude of the S-wave scattering process $D_s^+\pi^0 \to D_s^+\pi^0$ in the $D_{s0}^*(2317)^+$ resonance region [see (15), (16), and Fig. 6]. Our numerical estimates for the $D_{s0}^*(2317)^+ \to D_s^+\pi^0$ decay width do not contradict those available in the literature.

(3) It is important that the constructed model expressions for the complex of mixed amplitudes with $I=0$ and 1 entirely satisfy the requirements of unitarity.

(4) Our approach is fully applicable to the description of the mixing of the $D_{s0}^*(2317)^+$ with the supposed resonance isovector state $T_{c\bar{s}}(2327)^+$ [34,35]. But here it is necessary to note that the two-humped spectrum of $\pi^+\pi^-$ in the decay of $D_{s1}(2460)^+ \to D_s^+\pi^+\pi^-$, which gave a hint about the existence of states $T_{c\bar{s}}(2327)$ with isospin $I=1$ [35], can be explained without their introduction [36].

(5) It would be interesting to extend such a phenomenological analysis to the isospin violating decay of $D_{s1}(2460)^+ \to D_s^{*+}\pi^0$. The presence of spin in particles in this case introduces certain complications into the calculations with a purely relativistic approach.

ACKNOWLEDGMENTS

The work was carried out within the framework of the state contract of the Sobolev Institute of Mathematics, Project No. FWNF-2022-0021.

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: THE FUNCTIONS $G_{ii}(s)$, $\dot{G}_{34}(s)$, AND $\hat{G}_{11}(s)-\hat{G}_{22}(s)$

The function $G_{ii}(s)$ is uniquely determined by the masses of particles a and b in the ith channel. Let $m_a > m_b$. Then for $i = 1$ $m_a = m_{D^0}$ and $m_b = m_{K^+}$ etc. Let $G_{ii}(s) = I_{ab}(s)$. The dispersion loop integral $I_{ab}(s)$ is given by for $s > m_{ab}^{(+)2}$

$$I_{ab}(s) = \frac{s - m_{ab}^{(+)2}}{\pi} \int_{m_{ab}^{(+)2}}^{\infty} \frac{\rho_{ab}(s')ds'}{(s' - m_{ab}^{(+)2})(s' - s - i\varepsilon)} = L_{ab}(s) + \rho_{ab}(s)\left(i - \frac{1}{\pi}\ln\frac{\sqrt{s - m_{ab}^{(-)2}} + \sqrt{s - m_{ab}^{(+)2}}}{\sqrt{s - m_{ab}^{(-)2}} - \sqrt{s - m_{ab}^{(+)2}}}\right), \quad (A1)$$

where $m_{ab}^{(\pm)} = m_a \pm m_b$, $\rho_{ab}(s) = \sqrt{s - m_{ab}^{(+)2}}\sqrt{s - m_{ab}^{(-)2}}/s$, and

$$L_{ab}(s) = \frac{1}{\pi}\left(\frac{s - m_{ab}^{(+)2}}{s}\right)\frac{m_{ab}^{(-)}}{m_{ab}^{(+)}}\ln\frac{m_a}{m_b}; \quad (A2)$$

for $m_{ab}^{(-)2} < s < m_{ab}^{(+)2}$ $\rho_{ab}(s) = \sqrt{m_{ab}^{(+)2} - s}\sqrt{s - m_{ab}^{(-)2}}/s$ and

$$I_{ab}(s) = L_{ab}(s) - \rho_{ab}(s)\left(1 - \frac{2}{\pi}\arctan\frac{\sqrt{m_{ab}^{(+)2} - s}}{\sqrt{s - m_{ab}^{(-)2}}}\right); \quad (A3)$$

for $s < m_{ab}^{(-)2}$ $\rho_{ab}(s) = \sqrt{m_{ab}^{(+)2} - s}\sqrt{m_{ab}^{(-)2} - s}/s$ and

$$I_{ab}(s) = L_{ab}(s) + \frac{\rho_{ab}(s)}{\pi}\ln\frac{\sqrt{m_{ab}^{(+)2} - s} + \sqrt{m_{ab}^{(-)2} - s}}{\sqrt{m_{ab}^{(+)2} - s} - \sqrt{m_{ab}^{(-)2} - s}}. \quad (A4)$$

$$\dot{G}_{34}(s) = \frac{\Pi_{\pi^0\eta}}{m_\eta^2 - m_\pi^2}\left[G_{44}(s) - G_{33}(s) + \frac{1}{\pi}\left(\ln\frac{m_\eta}{m_{\pi^0}} + \frac{m_{D_s^+} - m_\eta}{m_{D_s^+} + m_\eta}\ln\frac{m_{D_s^+}}{m_\eta} - \frac{m_{D_s^+} - m_{\pi^0}}{m_{D_s^+} + m_{\pi^0}}\ln\frac{m_{D_s^+}}{m_{\pi^0}}\right)\right]. \quad (A5)$$

$$\hat{G}_{11}(s) - \hat{G}_{22}(s) = G_{11}(s) - G_{22}(s) + \frac{1}{\pi}\left(\ln\frac{m_{D^+}m_{K^0}}{m_{D^0}m_{K^+}} + \frac{m_{D^+} - m_{K^0}}{m_{D^+} + m_{K^0}}\ln\frac{m_{D^+}}{m_{K^0}} - \frac{m_{D^0} - m_{K^+}}{m_{D^0} + m_{K^+}}\ln\frac{m_{D^0}}{m_{K^+}}\right). \quad (A6)$$

Let us explain the origin of the sign in the relation $\ddot{G}_{33}(s) = -\ddot{G}_{44}(s) = -\dot{G}_{34}(s)\frac{\Pi_{\pi^0\eta}}{m_\eta^2 - m_{\pi^0}^2}$ [see the text below Eq. (8)]. The function $\ddot{G}_{33}(s)$ can be calculated in two ways, using either the right or left sides of the equality shown in Fig. 3, after replacing η with π^0, π^0 with η, and $I = 0$ with $I = 1$. The function $\dot{G}_{34}(s)$ is symmetric with respect to such a replacement [see (A5)]. Therefore, the right-hand side of this equality as a whole changes sign. The change in sign of its left-hand side after the indicated replacement can be verified by direct calculation.

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